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ELEMENTARY ARITHMETIC,

WITH BRIEF NOTICES OF ITS HISTORY.

SECTION I.

OF NUMBERS.

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INTRODUCTION.

NUMBERS.

It has been remarked, that one of the most important, and yet one of the most neglected branches of every science, is its history. The following brief notices of the history of the science of number make no pretence to completeness. If they invest the subject with interest to the mind of the intelligent student, and lead him to further inquiries, the object of the writer will have been answered.

In what age of the world the science of number had its origin—who first devised the method of counting by tens—who first invented symbols of notation, and separated the idea of number from the qualities of objects with which it was associated, are questions more easily proposed than answered satisfactorily.

It is highly probable that the origin of number was coeval with the origin of spoken language, and that, long before figures were invented, some rude methods of reckoning were devised, at first limited, but afterwards extended and improved as the wants and necessities of human society increased. The classifying by pairs would seem to suggest the simplest mode of reckoning. The counting by fives was probably the next step in numeration, and the practice of numbering by the five fingers on the two hands was the origin of counting by tens, as almost all children may be observed to do in their first efforts in counting. In the oldest writings which have been preserved to modern times, there is found a full recognition of this principle of counting by tens, tens of tens, tens of hundreds, and so on. Language still betrays by its structure the original mode of proceeding, and it is probable that the primitive words denoting numbers did not exceed five.

It was by abstracting or separating the idea of number from the ideas of the qualities of the things themselves, and expressing this abstraction in language, that the names of numbers have arisen, and the names of numbers being thus separated, could afterwards be applied to things with other qualities. The information, however, which can be collected from what remains on this subject, is both scanty and unsatisfactory. Some ancient languages recognised a dual number in the names of things, and the English words *pair* and *brace* are employed not universally, but only to some particular things; the same remark may be made on the word *leash*, applied to three particular things.

It is uncertain whether the earliest forms of written language were hieroglyphical or alphabetical, whether the letters denoting elementary sounds were formed from hieroglyphical characters; it is, however, certain that the initial letters of the names of numbers were in very early times employed as symbols of numbers. The brief notices here given of the early history of numbers, will be restricted to those peoples who have chiefly contributed by their discoveries and writings to our civilisation and advancement in knowledge.

In the fifteenth section of his Problems, Aristotle puts forth the following questions, touching the opinions held by philosophers of his time, as to the origin of counting by tens:—

"Why do all men, barbarians as well as Greeks, numerate up to ten, and not to any other number, as two, three, four, or five,[1] and then repeating one and five, two and five, as they do one and ten, two and ten, not counting beyond the tens, from which they again begin to repeat? For each of the numbers which precedes is one or two, and then some other, but they enumerate however, still making the number ten their limit. For they manifestly do it not by chance, but always. The truth is, what men do upon all occasions and always, they do not from chance, but from some law of nature. Whether is it, because ten is a perfect number? For it contains all the species of number, the even, the odd, the square, the cube, the linear, the plane, the prime, the composite. Or is because the number ten is a principle? For the numbers one, two, three, and four when added together produce the number ten. Or is it because the bodies which are in constant motion, are nine? Or is it because of ten numbers in continued proportion, four cubic numbers[2] are consummated, out of which numbers the Pythagoreans[3] say that the universe is constituted? Or is it because all men from the first have ten fingers? As therefore men have counters of number their own by nature, by this set, they numerate all other things."

Besides the idea of the division of numbers by tens, the names of the first ten numbers as they have descended to modern times are suggestive of questions for consideration to the student. The following list contains the names of the first ten numbers as preserved in seventeen languages, some of them being no longer spoken:—

1. Hebrew: echad, shnayim, shlosha, arbaa, khamisha, shisha, shiva, shmona, tisha, asara.
2. Arabic: wahad, ethnan, thalathat, arbaat, khamsat, sittat, sabaat, thamaniat, tessaat, aasherat.
3. Syriac: chad, treyn, tlotho, arbo, chamisho, shitho, shavo, tmonyo, tesho, eesro.
4. Persian: yak, du, sih, chahar, panj, shash, haft, hasht, nuh, dah.
5. Sanscrit: eka, dwi, tri, chatur, panchan, shash, saptan, ashtan, novan, dasan.
6. Greek: ἕις, δύο, τρεῖς, τέσσαρες, πέντε, ἕξ, ἑπτὰ, ὀκτὼ, ἐννέα, δέκα.
7. Latin: unus, duo, tres, quatuor, quinque, sex, septem, octo, novem, decem.
8. Italian: un, due, tre, quattro, cinque, sei, sette, otto, nove, dieci.
9. Spanish: uno, dos, tres, quatio, cenco, seis, siete, ocho, nueve, diez.
10. French: un, deux, trois, quatre, cinq, six, sept, huit, neuf, dix.
11. Welsh: un, dau, tri, pedwar, pump, chwech, saith, wyth, naw, deg.

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[1] This refers to the quinary scale of notation, instances of which are found in Homer, Odys. iv. 412; in Æschylus, Eumen. 738, and in other Greek writers.

[2] In Euc. viii. 10, it is demonstrated that if, beginning with unity, ten numbers are formed in continued proportion, four of these numbers will be cubic numbers.

[3] The Pythagorean philosophers indulged in fancies the most absurd, in the extraordinary powers they attributed to numbers; and among other absurdities they maintained that, of two combatants in the Games, the victor would be that man the letters of whose name, numerically estimated, expressed the greater number. In later times they were fond of forming words so that the numeral value of the letters should be equal to the same number, and there is an instance in the Greek Anthology (vol. ii., p. 412, Jacobs) in which a poet has applied the idea to describe a pestilent fellow. Having observed that the letters of his name *Δαμαγόρας* (mob orator) and *λοιμὸς* (pestilence) denoted, in the Greek notation, the same number, the following epigram declares, that when weighed in the balance, the latter was found to be the lighter.

Δαμαγόραν καὶ λοιμὸν ἰσόψηφόν τις ἀκούσας
Ἔστησ' ἀμφοτέρων τὸν τρόπον ἐκ κανόνος.
'Εις τὸ μέρος δὲ καθείλκετ' ἀνελκυσθὲν τὸ ταλάντον
Δαμαγόρου, λοιμὸν δ' εὗρεν ἐλαφρότερον.

12. Gaelic: aou, da, tri, ceithar, koig, sia, seachd, ochd, nai, deich.
13. Erse: aen, da, tri, keathair, kuig, se, secht, ocht, noi, deich.
14. Mœso-Gothic: ains, twai, thrins, fidwor, fimf, saihs, sibun, ahtan, nihun, taihun.
15. High German: ein, tue, thri, fiuuar, finfe, sehs, sibun, ohto, niguni, tehan.
16. Anglo-Saxon: an, twa, threo, feower, fif, six, seofon, eahta, nigon, tyn.
17. English: one, two, three, four, five, six, seven, eight, nine, ten.

On examining and comparing these names of the first ten numerals, it will be apparent that in some there is a complete or partial identity, and in others a diversity with more or less resemblance.[1] The use of the same or somewhat similar sounds to express the same ideas by the successive generations of men, suggests the high probability that they had a common origin, while the diversities are such as might arise from some confusion at a very remote period in the original language. The resemblances and diversities are too numerous, both in the numerals and in other fragments of elementary names, to be regarded as merely accidental. And considering the remote period alluded to, being above four thousand years, the mind is naturally led to the conclusion that the fragments of these primeval names are derived by each language from one of its cognates, or by all from one common source.

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[1] The close relation of the English names of the first ten numbers with those of the Anglo-Saxon, High German, and Mœso-Gothic is obvious. With respect to the names of numbers greater than ten, it may be remarked that the word *eleven*, Anglo-Saxon, *endlufon*, signifies *leave one* (that is above ten) being derived from *ein*, one, and the old verb *liben*, to remain. The word twelve is of like derivation, and means *leave two.*

The words *thirteen*, *fourteen*, &c., to *nineteen*, are formed from—three and ten, four and ten, &c.

The word *twenty* is derived from the High German *twentig*, *bis decem*, or from the more distant Mœso-Gothic *twaintegum*. In the same way are formed thirty, forty, &c., to ninety.

Hundred is a form of the Low German *hundert*, and is related to the High German and Anglo-Saxon *hund.*

Thousand: Anglo-Saxon, *thusend*, German, *tausend*, from the Mœso-Gothic, *tigos hund*, or *taihuns hund*, ten times a hundred.

Million comes from the Italian *millione.* The introduction into Italy of the Indian figures brought in a knowledge of numbers which neither the Latin nor the Italian language had names to express. This circumstance rendered some additions to the names of number necessary. The word *millione* has its origin in the Latin *mille*, and by the analogy of the Italian language the word *millione* means a great thousand, or, in a numerical sense, a thousand thousands. The units of the higher orders, *billione*, *trillione*, &c., are obviously formed from the word *millione*, with the Latin *bis*, *tris*, &c., prefixed, and thus forming a series of numerical words, of which each succeeding term is a million times that which immediately precedes it.

The numerical language of the Italians proceeding by *thousands* and by *millions* led to the custom of dividing numbers into periods of three figures and of six figures, and this mode of numeration being adapted to most of the languages of Europe, came into universal use with the terms million, billion, &c., borrowed from the Italian. Bp. Tonstall in his work "De Arte supputandi," published in 1522, speaks of the word *million* as a word in common use, and Dr. Robert Recorde, in his "Grounde of Artes," published in 1542, employs the word without any further remark than explaining its meaning, and dividing numbers into periods of three figures. The French system of numeration differs from the English by making the billion equal to a thousand millions, a trillion, a thousand billions, and so on.

The name *cipher* (τζφιρα) is borrowed from the Arabic *tsaphara*, which means blank or void, and is identical with the Sanscrit word *sunya.* In the Sanscrit notation the cipher was denoted by a point or by a small circle, which latter it appears from Planudes was preferred. The word cipher (Italian *zifra*, French *chiffre*) has several equivalents in use, as *nothing*, *nought*, *zero.* It was written *zefro* by the Spanish Moors, and might easily be changed into *zerro* or *zero*, at the time when the notation was translated from the Arabic by Spanish Moors and Jewish merchants.

The word cipher, from its importance in the system, has received a more extended meaning than its original sense. All the nine digits have been subjected to the general name of ciphers, from which the verb *to cipher* has been formed, having the same sense as *to calculate* with these figures.

Little is known respecting the origin and the early history of arithmetic of the ancient Hebrews or Syrians.[1] It has been conjectured that they were indebted to the Phœnicians, their neighbours, for what they knew of the art of numbering. The most ancient books—the writings of Moses—afford no evidence of the use of any numerical system of notation. In the text of the writings of Moses all numbers are expressed in words at length, and the counting is made by tens, hundreds, &c.

It is clear from the second chapter of the Second Book of the Chronicles that the Hebrews had commercial intercourse with the Phœnicians above a thousand years before the times of the Messiah. And the ancient tradition of the Greeks also tends to favour the opinion that Cadmus [קדם], a man from the East, was the first who introduced the use of letters into Greece from the Phœnicians. And it may be added that Proclus, in his Commentary on the First Book of Euclid's Elements, states that the Phœnicians, by reason of their traffic and commerce, were accounted the first inventors of arithmetic.

The ancient Hebrew and Samaritan alphabets consisted of twenty-two letters, and were employed to denote the nine digits, the nine tens, and the first four of the nine hundreds. The remaining five hundreds were represented by combining the symbols of the first four hundreds. In later times the final *caph*, *mem*, *nun*, *pe*, *tsadi* were added to make up the nine simple characters for the hundreds. All other numbers were expressed by placing together the simple characters denoting the component numbers required to make up their amount, with some few exceptions. The number 15 is denoted by טו, or 9 and 6, and not by יה, 10 and 5; because יה, Jah, being one of the names of God, it was imagined that such a use of the name would infringe the third commandment. For the same reason, perhaps, the number 1030 was not expressed by the characters אל, which form another of the names of the Deity.

The following are the characters of the Hebrew and Greek Alphabets, as they are applied to denote numbers :—

	1	2	3	4	5	6	7	8	9	
Units	א	ב	ג	ד	ה	ו	ז	ח	ט,	Hebrew.
	α	β	γ	δ	ε	ϛ	ζ	η	θ,	Greek.
Tens	י	כ	ל	מ	נ	ס	ע	פ	צ,	Hebrew.
	ι	κ	λ	μ	ν	ξ	ο	π	ϟ,	Greek.
Hundreds	ק	ר	ש	ת	ך	ם	ן	ף	ץ,	Hebrew.
	ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ,	Greek.

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[1] In the third volume of the new series of the Journal of Sacred Literature, Dr. W. Wright, the Professor of Arabic at Cambridge, has explained in his notice (pp. 128—130) of the Anecdota Syriaca of Dr. Land, a system of arithmetical notation employed in many of the oldest Syrian manuscripts not later than the ninth century. There are simple characters to denote 1, 2, 5, 6, 10, 20, 100; and these appear to have been combined to express other numbers, in some respects like the Roman notation.

A comparison of the Greek with the Hebrew numerical letters will suggest at once their common origin, and that the alphabets of both languages are derived from the same source, or that one is derived from the other. The fact of the correspondence of the order, the powers, and the names of the letters of both alphabets, is an argument in favour of this opinion. But the difference in form of the characters presents a difficulty, and it is uncertain whether the Greeks derived their numeral system from the same source as their alphabet. The system of the Greeks possesses an interest which does not attach either to that of the Romans or of the Hebrews, both from the improvements and extensions it received. A knowledge of these is essential for every one who may wish to read and understand the mathematical and astronomical writings of the Greeks.

The founders of the Roman name in Italy appear to have derived their descent from a colony of Pelasgi, who transported their language thither in its earliest and rudest form. For many ages, among a people incessantly occupied with war and conquest, their dialect continued almost unchanged till after the Punic wars, and the structure of the Latin language carries us back to a period anterior to any distinct vestige of the Greek language. The names of numbers must necessarily have been formed before any regular system of abbreviated numerical notation could have existed. There are no ancient writings extant which afford any satisfactory account of the origin of the Roman numerical symbols. From the large existing remains of Roman literature, besides monumental and other inscriptions, there is ample evidence of their universal use wherever the Roman arms and Roman language prevailed. Niebuhr informs us (vol. i. p. 134) that "what we call the Roman numerals are Etruscan, and they frequently occur on their monuments. They are remnants of a hieroglyphical mode of writing, which was in use before the age of the alphabetical, and like the numerals of the Aztecans, they represent certain objects that were associated with particular numbers. They are indigenous, and belong to the time when the west was subsisting with all its original peculiarities, before it received any influence from Asia." Notwithstanding this opinion of Niebuhr, that the Etruscan signs of numbers existed prior to the age of alphabetical writing, it must be considered doubtful, as little beyond conjecture can be attempted in the absence of evidence with respect to their origin and primary meaning. It is scarcely possible to discover what alterations in the signs took place after their adoption, or what additions or substitutions were made, except so far as they appear in the inscriptions on ancient monuments. The Romans employed seven elementary characters, as the primary signs of number, whose values are successively increased fivefold and twofold, beginning with unity. This was probably suggested by the two hands and the five fingers on each hand; or, as it is expressed by Ovid, Fast. iii., 126,

> "Seu quia tot digiti per quos numerare solemus."

These seven characters are I, V or Λ, X, L, C, D or IↃ, M: denoting respectively 1, 5, 10, 50, 100, 500, 1000. Other numbers are expressed in different ways by means of these seven symbols. A symbol repeated two or three, &c., times, denotes the double or treble of its value, as III stands for 3, and XX for 20. The symbols I, X, C are in general found repeated not more than four times; but in inscriptions I is found repeated six times, as IɪɪɪIVIR, for sevir or sextumvir. V and L are not found repeated. A symbol of less value postfixed to one of greater value increases the greater by that value, as VI means 5 increased by 1; and LX, 50 increased by 10; but if prefixed, it diminishes the greater by that number, as IV stands for 5 diminished by 1; and XL, 50 diminished by 10. This mode of notation by deficit was peculiar to the Romans, and is in accordance with the forms of their numerical words. Instead of octodecim for 18, their writers use duodeviginti, and undeviginti instead of novendecim. The letter M, the initial of *mille*, denoting 1000; 2000, 3000, 4000, &c., were denoted by IIM, IIIM, IVM, &c.; and by placing a line over the symbols their value was increased a thousand times, thus: Ī, L̄, C̄, M̄, &c., denote respectively 1000, 50,000, 100,000, 1,000,000, &c.

The latest improvement in the Roman notation was devised at a late period for the expression of large numbers. The method of proceeding was perfectly analogical. Taking the symbol C for 100, and IↃ for 500, by postfixing Ↄ once, twice, thrice, &c., to IↃ; the symbols IↃↃ, IↃↃↃ, IↃↃↃↃ, &c., were assumed to denote 5000, 50,000, 500,000, &c, respectively; and by prefixing C once, twice, thrice, &c., to IↃ, IↃↃ, IↃↃↃ, &c., respectively, their values become doubled, and the symbols CIↃ, CCIↃↃ, CCCIↃↃↃ, CCCCIↃↃↃↃ, &c., denoted 1000, 10,000, 100,000, 1,000,000, &c.

The Roman numerals are incapable of any material improvement. They could serve to register numbers, but could not afford the slightest aid in performing numerical calculations. In fact, they never were employed for that purpose. In the calculations which their accountants (calculatores, rationarii) had occasion to make, they were obliged to have recourse to a mechanical process, employing pebbles or counters. A box (*loculus*) of pebbles (*calculi*), and a board (*tabula*) on which the pebbles were placed in rows, formed their instruments of calculation. (Hor. Lib. I. Sat. vi. 74.) The terms *calculate*, *calculation*, are closely related to *calculus*, and in their primary meaning had reference to counting by means of pebbles. The board on which arithmetical operations were performed was also called abacus, and was divided from the right to the left by lines or grooves, on which the pebbles were placed to denote units, tens, &c. The operations of the abacus were rendered more commodious by substituting small beads strung on parallel threads, and sometimes by pegs stuck along grooves. With such an instrument, it is not difficult to perceive how addition and subtraction might be performed with ease and expedition, but to perform multiplication and division must have been a work of tedious labour.

The Roman notation was employed throughout the extent of the Empire for recording all accounts, whether fiscal or mercantile, and continued in use in Europe after its dissolution. The Roman notation was sanctioned by the almost universal employment of the Latin language in all subjects of literature and science, which for ages continued to be the language of the learned. It was employed in England from the time the Romans held possession of the island, and during the rule both of the Saxon and the Norman. The accounts in the Domesday Book are registered in the Roman characters. The dotation of the colleges of Oxford and Cambridge, and the annual accounts, were all recorded in the Roman characters long after the Indian notation had been introduced. It still continues to hold a position of considerable importance in the recording of dates, indexes, and monumental inscriptions in Western Europe.

In the early period of the history of the Hellenic race, before their wants and necessities called for the use of large numbers, the initial letters of the names of numbers were employed to express the numbers themselves. Thus, the letters, I, Π, Δ H, X, M, being the initial letters of the words Ἴος (for ἑις), Πέντε, Δέκα, Ἑκατὸν, Χίλιοι, Μύριοι, were employed to express 1, 5, 10, 100, 1000, 10,000. The other numbers were expressed by repeating or combining these six characters. Abbreviated combinations were also employed; as when any of these numeral letters were written within the capital letter Π, they denoted a number five times as great, as Δ written within Π denoted 50. This method was probably the first step towards a system of numerical notation, and, except for inscriptions, was superseded for the more perfect system formed with the letters of the alphabet. At a subsequent period the Greeks employed the letters of their alphabet with three supplementary symbols to express the first order of digits 1 to 9; the second, 10 to 90; and the third, 100 to 900. The fourth order of thousands was formed by subscribing an ι to each of the units of the first order. The fifth, sixth, &c., orders were formed by affixing M. or Μυ. for Μύριοι, 10,000, to each character of the first, second, &c., orders.

Of the three supplementary symbols employed in the Greek numerical notation, there being no letter corresponding to the Hebrew *vau*, the character ϛ is used for the number 6, and called ἐπίσημον βαῦ, indicating vau. The other two symbols were ϟ for 90, and ϡ for 900, the former called ἐπίσημόν κοππὰ and the latter ἐπίσημον σανπὶ, that is, indicating Koph, and indicating Tsadi. By combining these symbols any other numbers could be expressed as οα denotes 41; υα, 401; δ͵α, 4001; α͵σλδ, 1234; λζ.Μυ, 370,000. Neither the order nor the number of the characters had any effect in fixing the value of any number intended to be expressed. The value of the same combination of symbols is the same in whatever order they are placed; in general, however, they were written from right to left, according to their increasing value.

The oldest arithmetical writings of the Greeks which have descended to modern times are to be found in the seventh, eighth, ninth, and tenth books of the Elements of Euclid, who lived between B.C. 323 and 284. These four books contain complete treatises on numbers, their properties, proportions, commensurable and incommensurable, and their application to geometry. Diophantus, who has been placed by some writers as early as A.D. 280, by others as contemporary with the Emperor Julian, was the author of thirteen books on arithmetic and algebra. Only six of the thirteen books are known to be extant. The first four were translated into French by Simon Stevin, and the other two by Albert Girard. The six books are printed in the collected works of Stevin, revised and augmented by his friend Albert Girard in 1634.

The progress of astronomical and mathematical science occasioned the necessity of larger numbers than could be expressed by the Greek notation then in use. Archimedes, who lived between B.C. 287 and 212, improved and extended the Greek method of notation. In his work entitled ψαμμίτης or Arenarius, he proposed to express a number which should exceed the number of the grains of sand that might be contained in the sphere of the universe as conceived by Aristarchus. He assumed a scale of numeration whose radix or base is a myriad of myriads, or ten thousand times ten thousand, the number which then formed the limit of the Greek numerical language. All numbers less than this radix he called primary numbers, and the radix itself he made the unit of secondary numbers; he then proceeds to ternary, quaternary, and other numbers of higher orders, forming successive classes.

According to the Indian notation, the units of Archimedes, the primary, secondary, ternary, quaternary, quinary, &c., orders of numbers, will consist of 1; 1 with 8 ciphers; 1 with 16 ciphers; 1 with 24 ciphers; 1 with 32 ciphers, &c., respectively. And the unit of the ten thousand times ten thousandth order would consist of 1 with 799999992 ciphers.

His method for determining the number of places in any required number is the following. He supposes a series of numbers beginning with unity, in continued proportion, and shows that the product of any two terms of this series is equal to that term whose place reckoned from the first, is less by unity than the sum of the two numbers which indicate the places of the two terms; as the 7th term of the series is equal to the product of the 3rd and 5th terms. He then assumes the series 1, 10, 100, 1000, &c., in which each successive term increases tenfold, or where the common ratio is 10. The first 8 terms of this series (omitting the first term) are primary numbers; the next 8 terms, secondary numbers; the third 8, ternary numbers, and so on; and the question is to determine that term in the series which is equal to the product of any two assigned terms, or the term whose place is the sum (less by unity) of the two numbers which indicate the places of the two assigned terms. The classes themselves he calls *octades*, or periods of eight, from each class requiring eight symbols, or eight places of figures of common notation, to express the numbers included in each class. He then shows, without finding or assuming the number itself, that the number requiring for its expression not more than eight of these octades, or, in our notation, not exceeding 64 places of figures, will exceed the number of the grains of sand in the sphere of Aristarchus, each grain of sand being so small that 10,000 of them are less than one seed of poppy.

Apollonius, about 240 B.C., adopted the plan of Archimedes of classifying numbers, but instead of the *octades* of Archimedes, he adopted *tetrads*, reducing the radix from ten thousand times ten thousand, to ten thousand. The units after the first class he designated in order, the single myriad, the double myriad, the treble myriad, &c., which he denoted by $M\alpha$, $M\beta$, $M\gamma$, &c., respectively. His chief object, however, appears to have been to simplify the process of multiplication, and to make the multiplication of all higher numbers dependent on the product of any two of the first nine digits. The work of Apollonius has perished, and even the name of its title is unknown. It is highly probable that the substance of it was embodied in the first two books of the mathematical collections of Pappus. Only a fragment of the latter portion of the second part is known to be extant, in which are exhibited several examples of the method of Apollonius. This fragment was published by Dr. Wallis with a Latin translation and notes in 1688, and is reprinted in the third volume of his works, pp. 595—614.

It seems obvious that had Archimedes and Apollonius proceeded on the same principle for all numbers, their arithmetic would have been greatly improved. And in expressing a number of significant figures, it is strange that it was not perceived that no subscribed marks would be necessary, as the position of the symbols alone would be sufficient to indicate what orders of numbers the symbols were intended to represent. The only difficulty that could arise would be when all the different orders were not entirely filled up. This difficulty in the Indian method has been overcome by the introduction of the character 0, whose numeral value is nothing by itself, but which serves to keep every other symbol in the place of order it is intended to occupy. It is extraordinary that this should have remained altogether undiscovered by so acute a people as the Greeks, when its necessity seems to arise so naturally from the case itself.

In an article by Mr. Whish in the first part of the Transactions of the Literary Society of Madras,[1] "On the Alphabetical System of the Hindus," mention is made of a mathematical tract named after its author Arya-Bhatta. This tract contains a very ancient system of numerical notation by means of the letters of the Sanscrit alphabet. Mr. Whish states that he has been unable to discover whether the notation had been in use before the time of the author, or whether he invented it himself, and adds that he is not aware of its use among mathematicians, not having found it in their works. He remarks that the Lilavati is composed from the second chapter of the tract.

Mr. Whish also notes another system which has from time immemorial prevailed among the learned in the southern part of India, and from an astrological work shows that this system was in existence nearly 2000 years ago. He concludes his article in these words:—"Suffice it to say, that even though the decimal scale of arithmetic has existed from time immemorial in India, it is still well ascertained that an alphabetical notation has been in use 1600 years ago in Northern India, the date of its origin cannot probably now be fixed; and that a totally different system of notation by the letters of the same alphabet, has been in use in the South of India, the period of whose origin, though its existence is traced back for near two chiliads of years, is also indeterminable." This subject engaged the attention of the late Mr. Princep, who, by means of the inscriptions and the coins of ancient Surashtra, informs his readers that he "has been fortunate enough to light upon a clue to the ancient forms of the Sanscrit numerals."[2] He declares that the most ancient mode of denoting number in the Sanscrit language, was by the use of letters in alphabetical order, but is unable to state when this system was exchanged for that of the decimal notation, and adds that the Nagari numerals extant on numerous monuments of the ninth and tenth centuries do not materially differ from those at present used in India.

From some grants recorded on plates of copper, and supposed to be of the third century, Mr. Princep discovered that some letters denote figures which express the dates of the grants, and he considered that he might venture to assign six of the nine digits.

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[1] "Transactions of the Literary Society of Madras." "On the Alphabetical Notation of the Hindus." London, 1827.

[2] "Essays on Indian Antiquities," by J. Princep, F.R.S., edited with notes and additional matter by E. Thomas, late of the Bengal Civil Service. 2 vols., 8vo. 1858. "Essay on the ancient Sanscrit Numerals," art. xix., vol. ii.

On some Surashtra coins, besides a legend in corrupted Greek characters, he observed a few strange marks, which he found to be numerals of the same form and equal variety as those on the copper-plate grants. He also remarks that there are varieties in some of the forms of the nine numerals; and besides that, in many of the ancient systems, separate symbols were used to denote 10, 20, &c., in combination with the nine units severally.

Mr. Princep further states in his essay, that he was in hopes of tracing the ancient Sanscrit symbols of number by a comparison of them with the numeral systems of those Indian alphabets which have most resemblance to the forms of the earliest centuries. In a plate he exhibits the forms of the numeral letters of several alphabets, and remarks: "Upon regarding attentively the forms of many of the numerals, one cannot but be led to suppose that the initial letters of the written names were, many of them, adopted as their numerical symbols." This similarity may clearly be seen in the forms of some of the Sanscrit numerals, and his editor observes: "It is possible that the new data, which has lately become available, may contribute materially to solve the general problem of the system under which the ancient Indian scheme was primarily conceived."

The oldest treatise known to be extant on decimal arithmetic forms the twelfth chapter of a System of Astronomy written by Brahmegupta in India. Mr. Colebrooke has determined with great probability that this writer flourished in the sixth or at the beginning of the seventh century of the Christian era, and antecedent therefore to the cultivation of those sciences by the Mohammedans. It is written in Sanscrit verse, and comprises twenty-one chapters. The twelfth treats of arithmetic, and is divided into ten sections: Algorithm, Mixture, Progression, Plane Figure, Excavations, Stacks, Saw, Mounds of Grain, Measure of Shadow, Supplement.

There is also a perpetual commentary on the whole work. Each verse of the text is quoted at length, and interpreted with elucidations and remarks. It is moreover certain that this treatise on arithmetic was not the first written in Hindustan. It is impossible to doubt of the knowledge of these sciences, and of their wide extension long before they assumed the form in which they are found in writing. The fact of a treatise on arithmetic being found in the midst of a system of astronomy, and the employment of the principles of one science used to aid in the development of another, afford a very strong presumption in favour of their existence in a previous age, and of their having passed through many stages of addition and improvement.

This treatise, though the most ancient known, is not so complete or extensive as the Lilavati,[1] another treatise on arithmetic, which constitutes the first part of Mr. Colebrooke's work. The Lilavati stands as one of the preliminary portions of a course of astronomy, by Bhascara. He informs his readers that his work is compiled and abridged from more diffuse works, and was completed at a period corresponding to A.D. 1150.

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[1] The oldest treatise on arithmetic possessed by the Hindus, the Lilavati, remounts no higher than the eleventh century of our era. This famous composition, to which the vanity and ignorance of that people claim a divine original, is but a very poor performance, containing merely a few scanty precepts couched in obscure memorial verses. The examples annexed to these rules, often written probably by later hands in the margin, are generally trifling and ill-chosen. Indeed, the Lilavati exhibits nothing that deserves the slightest notice, except the additions made by its Persian commentators. The Hindus had not the sagacity to perceive the various advantages to be derived from the denary notation. They remained entirely ignorant of decimal fractions, with which their acute neighbours, the Chinese, have been familiarly acquainted from the remotest ages. Their nume-

The Lilavati appears to have gained in past times great authority, and to have superseded the preceding treatises, and formed the subject of study in countries and places so remote from each other as the north and west of Hindustan and the southern peninsula. There is a Persian version of the Lilavati, which was undertaken by Faizi at the command of the Emperor Acbar, and was completed A.D. 1587.[1]

The occasion of writing the Lilavati, given by Faizi in the preface to his Persian translation, was rendered into English by Mr. Strachey, and printed in the introduction to Dr. Taylor's translation of the Lilavati. The account[2] thus proceeds :—

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rical operations are unnecessarily complicated, following closely the procedure which the application of an alphabet had obliged the Greeks to employ.—*Leslie's Philosophy of Arithmetic*, p. 225.

There is another Eastern people [the Chinese] remarkable at once for the great antiquity and unchangeable character of their existing institutions, who possess a numerical language of great extent, connected with a very perfect system of numeration. . . .

As the Chinese are not in the possession of the method of arithmetical notation by nine figures and zero, they clearly can have no proper claim to its invention, however nearly in some respects they may have approximated to it; for it is next to impossible that a system of numeration, so much more perfect and commodious than their own, if once generally known or practised, could ever have been lost or abandoned.—*Dean Peacock's History of Arithmetic in the Encyclopædia Metropolitana*, vol. i., pp. 375, 376.

[1] The following works on the Indian Arithmetic and Algebra have been translated into the English language :—

"Bija Ganita, or the Algebra of the Hindus," by Edward Strachey, of the East India Company's Bengal Civil Service. London, 1813.

"Lilawati; or, a Treatise on Arithmetic and Geometry," by Bhascara Acharya; translated from the original Sanscrit by John Taylor, M.D., of the Hon. East India Company's Bombay Medical Establishment. Bombay, 1816.

"Algebra, with Arithmetic and Mensuration," from the Sanscrit of Brahmegupta and Bhascara; translated by Henry Thomas Colebrooke, F.R.S. London, 1817.

[2] This account is not found in the Sanscrit copies of the Lilavati, nor in any of the commentaries on that work. The language of the following questions which appear in pages 5, 6, 21, 24, 31, of Mr. Colebrooke's translation from the Sanscrit, supplies some internal evidence for the truth of the story :—

"Dear, intelligent Lilavati, if thou be skilled in addition and subtraction, tell me the sum of 2, 5, 32, 193, 18, 10, and 100 added together; and the remainder, when their sum is subtracted from 10,000."

"Beautiful and dear Lilavati, whose eyes are like a fawn's, tell me what are the numbers resulting from 135 taken into 12."

"If thou be skilled in multiplication by whole or by parts, whether by subdivision of form or separation of digits; tell me, auspicious woman, what is the quotient of the product divided by the same multiplier."

"Pretty girl, with tremulous eyes, if thou know the correct method of inversion, tell me what is the number, which, multiplied by three, and added to three-quarters of the quotient, and divided by seven, and reduced by subtraction of a third part of the quotient, and then multiplied into itself, and having fifty-two subtracted from the product, and the square root of the product extracted, and eight added, and the sum divided by ten, yields two?"

"Out of a swarm of bees, one-fifth part settled on a blossom of Cadamba, and one-third on a flower of Silind'hri; three times the difference of these numbers flew to the bloom of a Cutaja; one bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a Jasmin and Pandanus. Tell me, charming woman, the number of bees."

"It is said that the composing the Lilavati was occasioned by the following circumstance. Lilavati was the name of the daughter of the author, Bhascara, concerning whom it appeared, from the qualities of the ascendant at her birth, that she was destined to pass her life unmarried, and to remain without children. The father ascertained a lucky hour for contracting her in marriage, that she might be firmly connected, and have children. It is said that when that hour approached he brought his daughter and his intended son near him. He left the hour-cup on the vessel of water, and kept in attendance a time-knowing astrologer, in order that when the cup should subside in the water, those two precious jewels should be united. But, as the intended arrangement was not according to destiny, it happened that the girl, from a curiosity natural to children, looked into the cup, to observe the water coming in at the hole, when by chance a pearl separated from her bridal dress, fell into the cup, and, rolling down to the hole, stopped the influx of the water. So the astrologer waited in expectation of the promised hour. When the operation of the cup had thus been delayed beyond all moderate time, the father was in consternation, and examining, he found that a small pearl had stopped the course of the water, and that the long-expected hour was past. In short, the father, thus disappointed, said to his unfortunate daughter, I will write a book of your name, which shall remain to the latest times: for a good name is a second life, and the groundwork of eternal existence."

There are several commentaries in Sanscrit extant on this work. The oldest on the Lilavati was composed about A.D. 1420. Another bears a date corresponding to A.D. 1545, and exhibits a copious exposition of the text of the Lilavati, with demonstrations of the rules. Frequent references in one of the commentaries on the Lilavati are made to Arya-Bhatta, who was regarded as the most ancient of their uninspired writers. He flourished not later than the fifth century, but probably as early as the third or fourth century of the Christian era.

It is highly probable he was the chief of those who improved and advanced the science of arithmetic to that state of perfection at which it has been nearly stationary in Hindustan for ages. The Hindus themselves can give no account of the origin or discovery of the science of arithmetic, nor even of the decimal notation, one of the most simple, and at the same time the most perfect of inventions. One of their commentators writes, that "the invention of nine figures with device of place to make them suffice for all numbers, is ascribed to the beneficent Creator of the universe." This opinion at least points to a period of great antiquity, probably antecedent to the existence of any written account of the invention.

La Place, in his great work, has expressed the following opinion of the Indian notation:—"The idea of expressing all quantities by nine figures whereby is imparted to them both an absolute value and one by position, is so simple, that this very simplicity is the reason for our not being sufficiently aware how much admiration it deserves. But it is this simplicity, and the facility which calculations acquire by it, that raises the arithmetical system of the Indians to the rank of the most useful inventions. How difficult it was to discover such a method may be inferred from the circumstance that it escaped the talents of Archimedes and Apollonius of Perga, two men of the most profound genius of antiquity."

The Sanscrit language has independent names for the first seventeen orders of units in the decimal scale, 1, 10, 100, &c., the seventeenth being the place of hundreds of thousands of billions in the English mode of reckoning, which has only independent names for the first five — namely, one, ten, hundred, thousand, million. So large a vocabulary of names of the orders of units has no parallel in any other language, ancient or modern; and this fact affords a strong presumption of the high antiquity of the system of notation by nine figures with "device of place." In any number consisting of more figures than one, the place where no figure belongs to it was shown by a blank, and to obviate mistake was denoted by a dot or a small circle.

The stern and desolating wars which followed the rise of the Mohammedan power in the seventh century, extended both to the east and the west. Egypt, the chief seat of learning, was invaded and conquered A.D. 640, and the famous library of Alexandria was destroyed. The rapid conquests of the Arabian armies soon led to the foundation of a powerful empire. The second Khalif Almansur ascended the throne A.D. 753, and shortly after transferred the seat of his government from Damascus to the newly-founded city of Bagdad.

The Arabians became conversant with the arithmetical and astronomical science of the Hindus long before they had any knowledge of the writings of the Greek mathematicians. The earliest notice is referred to the second century of the Hegira, about A.D. 773, in the reign of Almansur, when an Hindu astronomer visited his court at Bagdad, bringing with him a book of astronomical tables. The Khalif committed the work to Alfazari to be translated, and to be published "for a guide to the Arabians in matters pertaining to the stars." This version was afterwards known by the name of "the Greater Sindhind," and was in general use until the time of Almamun.

Haroun Alrashid, the grandson of Almansur, before his accession to the Khalifat, had overrun the provinces of Asia Minor and advanced as far as the Hellespont. The reigns of Alrashid and his successor, Almamun, were distinguished at Bagdad by the highest degree of luxury and splendour, which are displayed in many scenes of the famous tales of the Arabian Nights' entertainment. Almamun ascended the throne of the Khalifs A.D. 813.

It was the glory of his reign that he invited learned men from different countries for the introduction of science and learning into his dominions. Under his auspices and encouragement Arabic translations of Hindu and Greek science were undertaken. The few manuscripts of the philosophical writings of the Greeks which had escaped general ruin were diligently sought for and translated into Arabic, and Arabic commentaries were written to elucidate and explain these writings. By the desire of Almamun, before his accession to the Khalifat an abridgment of the Greater Sindhind was made by Mohammed Ben Musa, which was thenceforward known by the title of the Less Sindhind. That the Arabians derived their knowledge of astronomy and arithmetic from the Hindus is asserted with the universal consent of all Arabian authors from the time of Mohammed Ben Musa. It is well known from various works in Arabic that before the end of the tenth century the nine figures named *Hindasi*, from the country whence they had been derived, were in general use.

This concurrent history of Arabic writers is also further confirmed by the fact that the Arabians wrote their figures from left to right, after the manner of the Hindus, but contrary to the order of their writing, which was from right to left.

Under the reign of Almamun, Ptolemy's great work on astronomy and arithmetic was translated into Arabic, with the title of Almagest, a word formed from the Arabic article, and one of the words of σύνταξις μεγίστη, the title of Ptolemy's work. The arithmetics of Diophantus were translated into Arabic by Buzjani, in the fourth century of the Hegira, nearly two centuries after the Arabians had become acquainted with the arithmetic and the astronomy of the Hindus. Among the writers subsequent to Mohammed Ben Musa was Abulfaraj, the author of a treatise on computation. He lived in the twelfth century, and notices a work on numerical computation which Mohammed Ben Musa amplified, and is described as "a most expeditious and concise method, and testifies the ingenuity and acuteness of the Hindus."

Before the end of the eleventh century the Saracens had extended their conquests along the northern parts of Africa, and at an earlier period had established a flourishing kingdom in the southern provinces of Spain, which existed for upwards of seven centuries, until the reign of Ferdinand and Isabella in 1491, when Granada was taken, and the Saracen power in Spain came to an end. During the rule of the Saracens the arts and sciences of the East were cultivated and promoted, and the schools of the learned in Spain were in high repute in those early times.

In the latter part of the tenth century, Gerbert, a Benedictine monk, of Aurillac in Auvergne, is reported to have travelled into Spain, and there to have acquired a knowledge of the sciences of the Saracens, and also of the Arabic numerals. He was without doubt one of the remarkable men of his age, but how far he promoted or assisted in the extension of the knowledge of the Arabic arithmetic is not made out satisfactorily.

William of Malmesbury writes (De Gestis Anglorum) of Gerbert: "Abacum certe primus a Saracenis rapiens, regulas dedit, quæ a sudantibus abacistis vix intelliguntur." Gerbert afterwards became Archbishop of Rheims and of Ravenna, and in A.D. 999 was raised to the Popedom under the title of Sylvester II. He died in the year 1003. It is not improbable that other persons in their travels, both in Spain and in the East, acquired a knowledge of the Arabic numerals long before the use of them became general in the west of Europe. The intercourse of merchants in traffic, and of the hosts which, from the west of Europe, joined in the expeditions to the Holy Land during the Crusades, may also have afforded opportunities of gaining some acquaintance with the arts and knowledge of the Saracens.

The Arabic numerals are said to have been found in manuscripts of Spain of the eleventh and twelfth centuries. It is probable that much interesting information might be brought to light from any remains of the manuscript literature of the Saracens in Spain during these centuries. The Arabic numerals were certainly employed in the astronomical tables made by Alphonsus X., King of Castile, about A.D. 1252.

Leonardo of Pisa first made known from the Arabians the Hindu arithmetic and algebra in Italy. A manuscript of Leonardo's treatise bearing the title of "Liber Abbaci compositus a Leonardo filio Bonacci Pisano in anno 1202;" and a transcript of another treatise entitled "Leonardi Pisani de filiis, Bonacci . . . Practica Geometria composita anno 1220, " were found about 1750 by Targioni Tozzetti in the Magliabecchian Library at Florence, of which he had the care. In his preface to the Liber Abbaci, Leonardo relates that he had travelled into Egypt, Barbary, Syria, Greece, and Sicily. During his youth, at Bugia, in Barbary, where his father was scribe at the custom house for the merchants of Pisa who resorted thither, he there learned the Indian method of counting by nine figures. He states it to be more commodious than the methods used in other countries which he had visited; he therefore prosecuted the study, and, with some additions of his own, and some things taken from Euclid's Elements, he undertook the composition of his treatise that "the Latin race might no longer be found deficient in the complete knowledge of that method of computation." In the epistle prefixed to the revision of his Liber Abbaci in 1228, he professes to have taught the complete doctrine of numbers according to the Indian method.

The study of the Indian method of computation through the medium of Arabic in an African city having been introduced into Italy, the Italians were the first European people who cultivated this and its kindred sciences.

After the introduction of this new knowledge into Italy, numerous treatises were composed, and manuscript copies of the works of that age are found in the libraries of Italy and other parts of Europe. Villani, the earliest Florentine historian, writes of Paoli di Dagomari, who died about 1350, as a great geometer and most skilful arithmetician, and who surpassed both the ancients and moderns in the knowledge of equations. And Raffaelo Caracci, a Florentine arithmetician of the same century, wrote a work entitled "Ragionamento di Algebra," in which he speaks of Guglielmo di Lunis, who before his time had translated a treatise on Algebra from Arabic into Italian. This was most probably a translation of the Algebra of Mohammed Ben Musa, which Bombelli some years after spoke of as if it were well known in Italy.

Matthew Paris writes of John de Basingstoke:—"This Master John, moreover, brought into England the Greek numerical characters, and explained to his friends the knowledge and meaning of them. And this is chiefly remarkable of them, that every number is represented by a single character, which is not the case in the Roman numerals or in Algorithm."[1] It may be remarked that the word algorithm, as well as the word algebra, were, at that time, both new words from the Arabic, the former word being applied to the science of arithmetic with the Arabic numerals, and the latter to the generalised science of number.

John de Basingstoke was advanced to the Archdeaconry of Leicester by Robert Grosstete, Bishop of Lincoln, who was at that time a zealous promoter of Greek learning and of the sciences. He was the author of a work, "De Computo Ecclesiastico," and of the "Kalendarium Lincolniense," which latter was long held in high estimation. Copies of it still exist, of which only the Latin manuscript copies contain the Arabic numerals. After he was made bishop, Grosstete's firmness in resisting the encroachments of the papacy drew down upon him the censures of the Pope, but the Pope's censures had not the effect of inducing him to alter the line of conduct he deemed it his duty to his sovereign to adopt in the administration of his diocese. He died in the year 1253, at Buckden, the year before the death of John of Basingstoke.

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[1] Hic insuper magister Joannes figuras Græcorum numerales, et earum notitiam et significationes in Angliam portavit, et familiaribus suis declaravit. Per quas figuras etiam literæ repræsentantur. De quibus figuris hoc maxime admirandum, quod unica figura quilibet numerus repræsentatur; quod non est in Latino, vel in Algorismo.—*Matthew Paris.*

Johannes de Sacro Bosco, or John of Holywood or of Halifax as he is sometimes called, studied at Oxford. He was the author of a treatise on the Sphere, and of a tract, "De Arte Numerandi,"[1] both of which were celebrated works. This tract of Sacro Bosco gives the Arabic numerals, explains the local value, and gives the rules for arithmetical operations, including the rules for the square and cube roots. He was also the author of a work entitled "De Computo Ecclesiastico." His death took place at Paris, A.D. 1256.

Contemporary with Matthew Paris, Sacro Bosco, and Robert Grosstete, was Roger Bacon, a native of Ilchester, who was born about 1214, and died in 1292. He was one of the great men that held forth the light of truth in a dark age, only a few years after the kingdom of England had endured the degradation of a Papal Interdict of the third Pope Innocent. In his work entitled "Opus Majus" he highly commends the sciences, and that of number among them. He employs the word *algoristicus* several times, and repeats the names of the same set of rules as are given in the treatise of Sacro Bosco. The following extract from the first chapter of the "Opus Majus" can scarcely fail of being interesting to the student.

"There are four principal stumbling-blocks in the way of arriving at truth—*authority,*[2] *confirmed habit, appearances as they present themselves to the vulgar, and concealment of ignorance under the ostentation of*

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[1] In 1839 a small volume (pp. 120) was printed by Mr. Halliwell, with the title of "Rara Mathematica." In the collection will be found five tracts on the Arabic numerals. Of these the most important is the treatise of Sacro Bosco, the text of which he states is taken from a manuscript he purchased at the sale of the library of the Abbate Canonici at Venice.

There is another treatise entitled "Carmen de Algorismo," in hexameter verse, containing 255 lines. It appears that Alexander de Villa Dei was the author, and that he lived in the fourteenth century. The manuscript copies existing of this poem are very numerous, from which it may be inferred that it was highly valued and extensively read. There are manuscript copies of this treatise preserved both in the University Library and in the Library of Trinity College, Cambridge.

Mr. Halliwell quotes the following lines, which he has appended as a note, to the twenty-sixth line of the Carmen de Algorismo :—

En argorisme devon prendre
Vii especes . . . .
Adision subtracion
Doubloison mediacion
Monteploie et division
Et de radix enstracion
A chez vii especes savoir
Doit chascun en memoire avoir
Letres qui figures sont dites
Et qui excellens sont ecrites.
MS. Seld. Arch., B. 26.

These lines are probably as old as the time of Roger Bacon.

[2] Shakespeare saw the same causes at work in his day, and has left the record of his opinion of them in one of his sonnets in these words :—

"And Art made tongue-tied by Authority,
And Folly, doctor-like, controlling Skill,
And simple Truth miscalled Simplicity,
And captive Good attending captain Ill."

knowledge. The *authority* I mean here is that which many have violently usurped of their own self-will and with the lust of power, and to which the ignorant vulgar have yielded, to their own ruin, by the just judgment of God (in pernicionem propriam judicio Dei iusto). Now, where these obstructions exist, no reason can move, no judge decide, no law bind; right has no place, the dictates of nature no force; vice flourishes, virtue fades; truth expires, and falsehood rules supreme. Even if the first three can be got over by some great effort of reason, the fourth remains. Men presume to teach before they have learned, and fall into so many errors both in science and common life, that we see a thousand falsehoods for one truth. And this being the case, we must examine most strictly the opinions of our predecessors, that we may add what is lacking in them, and correct what is erroneous, but with all modesty and allowance. We must with all our strength prefer reason to custom, and the judgments of the wise and good to the opinions of the vulgar; and we must not use the triple argument—it is established—it is customary—it is common,—and therefore it is to be retained, whether in opposition to, or in accordance with the dictates of truth and reason." The man who held and expressed such opinions was a very dangerous person; and accordingly he was imprisoned, his works forbidden to be read, and his lectures prohibited in the University of Oxford. The learned monk, while engaged in his inquiries into the works of nature, and in his experiments in alchemy, was seriously believed to have practised magic, and to have had converse with evil spirits. So gross in that age was the ignorance of the clergy that even Anthony à Wood, the Oxford antiquary, who had no prejudices against the clergy, has stated with respect to their knowledge of geometry, that "they knew no property of the circle but that of keeping out the devil, and thought that the angles of a triangle would wound religion."

Thomas Bradwardine was born at Hatfield, in Sussex, about the close of the thirteenth century, and received his education at Merton College, Oxford. He was distinguished both as a divine and as a mathematician. He constantly attended Edward the Third during his wars in France, and was most probably present at the battle of Cressy in 1346. His works, "De Arithmetica Practica," and "De Proportionibus," were printed at Paris, the former in 1502, and the latter in 1495. He died in 1349, forty days after his consecration to the see of Canterbury.

Mabillon, in his work entitled "De Re Diplomatica," after the examination of above 6000 documents, writes that he found no authentic date in Arabic figures earlier than that of 1355, and that date in the handwriting of Petrarch.[1]

Geoffrey Chaucer, the poet, who died A.D. 1400, calls, in one of his poems, the Arabic numerals "the figures newe." He had visited Italy, where he would have learned that the science of number was there more cultivated than in England. At Florence, Venice, and other cities of Italy, both literature and the arts flourished in the thirteenth and fourteenth centuries, and the science of the arithmetic of commerce was both cultivated and improved by their extensive trade with other countries. He wrote a treatise on the Astrolabe in English for the use of his son. This is the earliest treatise in the English language written on any scientific subject. It has lately been edited by Mr. Skeat, of Christ's College, to which he has prefixed a very interesting preface. In pages xliii.—xlvi. will be found a table[1] of the fixed stars, copied from a manuscript which bears the date of 1223 in the Arabian characters, a date prior to that of Petrarch by more than a century. The character of the writing, however, is very much like that of manuscripts written in the fourteenth century. If these characters are really copied as written in the original manuscript, they constitute the earliest date as yet discovered in these characters.

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[1] In the "Journal of the Archæological Institute," vol. vii., p. 85, is a fac-simile of a date in a public document of 19 Edw. II., 1325 A.D., in which the date of the year is expressed in one part in Roman numerals, and in another in Arabic. The document is a warrant from Hugh le Despenser to Bonefez de Peruche and his partners, merchants of a company, to pay forty pounds, dated February 4, 19 Edw. II. In a different hand on the dorse, is a memorandum of the payment, written by one of the Italian merchants to whom the warrant was addressed.

They do not appear in the dates of the works of Caxton. In "The Myrrour or Ymage of the World," however, printed in 1480, where, treating of Arsmetryke, or Algorithm, among other sciences, he has given a woodcut of an arithmetician sitting before a desk, on which are tablets or papers marked with the nine figures. At St. Albans "The Myrrour of the World" was reprinted in 1506, in which the Arabic figures appear under the forms now in use.

The ancient calendars of the fourteenth and the early part of the fifteenth centuries, written before the invention of printing, supply some evidence of the manner in which a knowledge of the Arabic notation was generally made known both in England and in the countries of Europe.[2] Copies of these calendars are found in almost all the

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[1] "Tabula stellarum fixarum que ponuntur in Astrolabio certificata ad civitatem parisius cuius latitudo est 48 gradus et 30 minuta. In anno domini nostri Iesu Christi 1223."—Pref. p. xliii.—[*MS. Camb. Univ. Lib. Hh.*, 6. 8, *fol.* 236.]

[2] In Archbishop Parker's manuscript library, preserved in Corpus Christi College, Cambridge, there is a table of eclipses from 1330 to 1340, to which is subjoined a table, in three columns, containing the Roman and Arabic numerals, and another nearly the same as the Roman, but the characters different. The following explanation is subjoined :—"Omnis numerus vel omnis figura in algorismo primo loco se ipsum significat ; secundo loco, decies se ipsum significat ; tertio loco, centies se ; quarto loco, milesies se ; quinto loco, decies milesies se ; sexto loco, centies milesies se ; septimo loco, mille milesies se ; octavo loco, decies mille milesies se ; nono loco, centies mille milesies se ; decimo loco, mille milesies milesies se. Et sic multiplicando per decem centum et mille usque in infinitum computando versus sinistram."

The calendar of John Somers, of Oxford, written in 1380, was one of the most popular of the time, and the copies in general have this addition :—"Tabula docens algorismum legere, cujus utilitas est in brevi satis spatio numerum magnum comprehendere. Et quin numeri in Kalendario positi vix excedunt sexaginta, ultra illam summam non est protensa." There is a copy of this calendar in the British Museum, and several English translations among the manuscripts in the Ashmolean Library at Oxford.

Mr. Halliwell states in his "Rara Mathematica," that, in the year 1812, a small octavo volume was published at Hackney, containing an account of an almanack for the year 1386, probably one of the oldest in English. At this time the Indian notation appears to have been imperfectly understood, if one may judge from the mixture of Roman and Indian notations in numbers consisting of more than two figures ; as 52,220 is written thus—52 MCC 20.

There is a calendar preserved in the British Museum of about the date of 1403, which contains the numerals in the form they usually appear before the end of the fifteenth century.

There is another described in the "Archæologia," vol. xiii., p. 153, which public libraries of Europe, and some are preserved in the libraries of Oxford and Cambridge. In many of them is explained the Arabic method of notation, which would seem to show that the writers of the almanacks were using a notation requiring explanation, and differing from the Roman characters then in use. In the calendars of the latter part of the fifteenth century the explanations of the Arabic notation had ceased to appear.

Lucas Pacioli, or Lucas de Burgo, appears to have taught the sciences of algorithm and algebra at Venice about the year 1460, and to have noticed the names of men who had been his predecessors. In the year 1494 he published at Venice his "Summa de Arithmetica," the first work which was printed on the subject, and in 1525 a more complete form of it. This was his principal work. He professes to have consulted the earlier writers, Euclid, Sacro Bosco, Leonardo of Pisa, and others. The work itself treats of arithmetic, algebra, and geometry. In the first part he explains the properties of numbers and rules of the arithmetic of commerce, and gives an account of the principles of keeping merchants' accounts by double entry, afterwards called the Italian method. He also explains the rules of interest, exchange, barter, &c.

The "Summa de Arithmetica" was not printed till more than half a century after the invention of printing. The history of arithmetic and other sciences is almost entirely the history of books and manuscripts which treat on these subjects, with very little of contemporaneous history to show how extensive or otherwise was the knowledge of these sciences.

Cuthbert Tonstall was born in 1476, and studied first at the Universities of Oxford and Cambridge, and afterwards at Padua. Erasmus and Tonstall were firm friends. Sir Thomas More, in a letter to Erasmus, wrote of Tonstall: "As there was no man more adorned with knowledge and good literature, no man more severe and of greater integrity for his life and manners, so there was no man a more sweet and pleasant companion, with whom a man would rather choose to converse."

The work, "De Arte Supputandi," composed by Cuthbert Tonstall, was published in 1522. In his dedication to Sir Thomas More he professes to have read all the books that had ever been written on the subject. Professor De Morgan remarks: "This book was a farewell to the sciences, on the author's appointment to the See of London, and is decidedly the most classical which was ever written on the subject in Latin, both in purity of style and in goodness of matter. For plain common sense, well expressed, and learning most visible in the habits it had formed, Tonstall's book has been rarely surpassed, and never in the subject of which it treats."

Robert Recorde was educated at Oxford, and elected Fellow of All Souls' College in 1531, where he appears to have zealously promoted the study of the mathematical sciences. In 1545 he was admitted to the degree of M.D. at Cambridge, where he taught arithmetic and other parts of mathematical science. His published writings prove him to have been no common man, and he is thus acknowledged by the

contains the following account of the Arabic notation :—"Nota quod quælibet figura algorismi in primo loco signat se ipsam, et in secundo decies se. Tertio loco, centies se ipsam. Quarto loco, millesies se. Quinto loco, decies millesies se. Sexto loco, centies millesies se. Septimo loco, mille millesies se. Et semper incipiendum est computare a parte sinistra."

late Professor De Morgan:—"The founder of the school of English writers (to any useful or sensible purpose) is Robert Recorde, the physician, a man whose memory deserves a much larger portion of fame than it has met with on several accounts. He was the first who wrote on arithmetic in English (that is, anything of a higher cast than the works mentioned by Tonstall); the first who wrote on geometry in English; the first who introduced algebra into England; the first who wrote on astronomy and the doctrine of the sphere in English; the first, and finally the first Englishman (in all probability) who adopted the system of Copernicus."

Some of his works passed through numerous editions, and were long in estimation. His "Grounde of Artes" was first published in 1549, and dedicated to King Edward VI. The last edition of this work was published in 1669, with additions by Edward Hatton. "The Castle of Knowledge" was first printed in 1556, and dedicated in English to Queen Mary, and in Latin to Cardinal Pole. "The Whetstone of Witte," published 1557, was dedicated to "the Companie of venturers into Moscovia." In the "Pathway to Knowledge" he thus writes his opinion of the authority of Ptolemy:—

"No man can worthely praise Ptolemye, his travell being so great, his diligence so exacte in observations, and conference with all nations, and all ages, and his reasonable examination of all opinions, with demonstrable confirmation of his owne assertion, yet muste you and all men take heed, that both in him and in al mennes workes, you be not abused by their autoritye, but evermore attend to their reasons, and examine them well, ever regarding more what is saide, and how it is proved, than who saieth it; for autoritye oft times deceiveth many menne," &c.

He wrote on other subjects besides the mathematical sciences; one of these works, "The Urinal of Physic," may be named as having passed through four or five editions. It is melancholy to add that a man so learned and accomplished in various knowledge was imprisoned for debt in the King's Bench Prison, and died there, probably in 1588.

William Buckley, a native of Lichfield, was educated at Eton, where he was elected to King's College, Cambridge, in 1537, and was admitted M.A. in 1545. He was much esteemed by King Edward VI., and during his reign he was sent for by Sir John Cheke, when Provost of King's College, to instruct the students in arithmetic and geometry. He was the author of a small tract on arithmetic, entitled "Arithmetica Memorativa," written in Hexameter verse. It contains about 320 verses, giving all the ordinary rules of the science as then known and practised. It was first published in 1550, and afterwards printed at the end of "Seaton's Dialectica," at Cambridge, 1631.[1]

It is a fact, and one which exhibits the slow progress of the human mind, not only in discovery and invention, but even in the application of well-known principles, that the extension of the denary notation to the descending scale was not discovered before the latter part of the sixteenth century, more than a thousand years after the decimal arithmetic of integers had been known and cultivated in Hindustan. In the notation of integers it was well understood that each succeeding figure placed on the left had a value ten times as great as the figure next to it on the right, and if the converse of this had been perceived, it would have been obvious that the figure on the right of any other figure in the scale was one-tenth of the value of that adjacent to it on the left. But the truth is, it had not occurred to any one to apply this idea beyond the place of units, and thus to extend the scale to express decimal fractions. The scale thus extended would have caused no interruption of the law of continuity either in the ascending or descending parts of the scale, but would have rendered the scheme of notation complete for the expression of the smallest possible decimal fraction as well as the greatest possible integral number.

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[1] At the end is placed the following epigram:—

Περὶ τῆς χρείας τῆς ἀριθμετικῆς τέχνης.

Μέτρον τε, στάθμην τε, ζυγόν τε, καὶ ἶσον ἀριθμοὶ
  Εὗρον μέν πρῶτοι. τοὺς δὲ δικαιοσύνη.
Ἐν δὲ δικαιοσύνῃ συλλήβδην πᾶσ' ἀρετή 'στι.
  Ὅυ τι δίκαιος ἄρ' ὃυν πᾶς ἀνάριθμος ἀνήρ.

The earliest notice of decimal fractions is found in a small tract written by Simon Stevin, of Bruges, in Flemish, and published about 1590. He afterwards translated it into French, as he himself informs his readers. In the collection of his works published after his death for the benefit of his widow and orphans, by his friend Albert Girard, it will be found at the end of the treatise on Arithmetique. His tract describes the advantages of this extension of the descending scale of the denary notation, and calls decimal fractions "Nombres de disme." He designates the first, second, third, &c., places of decimals by the numbers 1, 2, 3, &c., placed in small circles,[1] reserving 0, included in the same manner for the place of integers. These characteristic marks in his tract are written after the figures whose places in the scale they severally mark; thus, 8(0)9(1)3(2)7(3), will signify $8,\frac{9}{10}$, $\frac{3}{100}$, $\frac{7}{1000}$, or $8\frac{937}{1000}$. In the same manner 3(4)7(5)8(6), mean $\frac{3}{10000}$, $\frac{7}{100000}$, $\frac{8}{1000000}$, or $\frac{378}{1000000}$. The characteristic figures are also found written in operations both above and below the figures they distinguish, according to convenience.

The notation of sexagesimals was continued to be used in astronomical calculations after the introduction of the Indian notation. The mode, also, of marking degrees, minutes, seconds, &c., was retained. It appears that Stifelius, in his "Arithmetica Integra," published in 1544, was the first who indicated minutes, seconds, &c., of the sexagesimal scale by the words minuta, prima, secunda, tertia, &c., and employed the small figures $\tilde{2}$, $\tilde{3}$, $\tilde{4}$, &c., with a circumflex to distinguish them. Thus in page 65 he writes: "$\overset{\text{grad.}}{1}\quad \overset{\text{min.}}{1}\quad \overset{\tilde{2}}{1}\quad \overset{\tilde{3}}{1}\quad \overset{\tilde{4}}{1}$ instead of $1^{0}\ 1^{\text{I}}\ 1^{\text{II}}\ 1^{\text{III}}\ 1^{\text{IV}}$, which signifies 1 degree, 1 minute, 1 second, 1 third, 1 fourth. It is not improbable that Stevin, from this mode of noting the orders of sexagesimals, was led to mark in a similar way the orders of decimal fractions.

The dedication of Simon Stevin's tract, as translated by Mr. Norton,[2] begins with the following passage:—"Many seeing the smalnes of this book, and considering your worthynes to whom it is dedicated, may perchance esteeme this our conceyte absurd. But if the proportion be considered, the small quantity thereof compared to humane imbecility, and the great utility unto high and ingenious intendiments, it will be found to have made comparison of the extreame tearmes, which permit not any conversion of proportion. But what of that? Is this an admirable invention? No, certainly; for it is so meane as that it scant deserveth the name of an invention. For as the countryman by chance sometime findeth a great treasure, without any use of skill or cunning, so hath it happened herein. Therefore if any will thinke, that I vaunt my selfe of my knowledge, because of the explication of these utilities, out of doubt, he showeth himselfe to have neyther judgement, understanding, nor knowledge to discerne simple things from ingenious inventions, but he (rather) seemeth envious of the common benefite; yet, howsoever, it were not fit to omit the benefit hereof, for the convenience of such calumny." This most important invention of Simon Stevin appears not to have been appreciated by his contemporaries, but some time elapsed before its excellence was perceived and its use discovered. The notation he employed, being analogous to the notation of sexagesimals then in use, was a needless and cumbrous addition, and this probably did not favour its general adoption. And it may be observed that this notation appears to have kept from his view, that the ratio of the descending scale from the unit's place towards the right was the inverse of the ascending scale from the unit's place towards the left, and that the perfection of his scheme only required some mark for separating the integers from the decimals.

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[1] Parentheses are substituted in the text instead of the small circles employed by Stevin, containing the figures which indicate the places of each decimal figure.

[2] The tract of Stevin, in 1608, was translated by Robert Norton into English, and published with the following title:—"Disme: The Art of Tenths; or Decimall Arithmetike; teaching how to performe all computations whatsoever by whole numbers without fractions, by the foure principles of common arithmeticke; namely, addition, substraction, multiplication, and division. Invented by the excellent mathematician, Simon Stevin. Published in English, with some additions, by Robert Norton, gent. Imprinted at London by S. S. for Hugh Astley, and are to be sold at his shop at Saint Magnus Corner. 1608." [4to, pp. 37.]

This important improvement was effected by John Napier, the inventor of logarithms. Instead of employing the notation of Stevinus, he simply separates the integers from the decimals by placing a point between them without making any remark on his own simplification of the notation. He afterwards writes the result of his example by placing one, two, three, &c., accents at the right of the first, second, third, &c., places of decimals. In page 21 of his "Rabdologia," which he published in 1617, he gives the following account in an "Admonitio pro Decimali Arithmetica":—"But should those fractions whose denominators are various be found disagreeable on account of the difficulty of working with them, and should that other kind, whose denominators are always tenth, or hundredth, or thousandth, &c., parts (which that most learned mathematician Simon Stevin, in his 'Decimal Arithmetic' notes and names in this manner, (1) firsts, (2) seconds, (3) thirds), be preferred on account of their effecting the same practical facility as integers, then the vulgar division being completed and concluded with a period or a comma, you can annex to the dividend or remainder one cipher for tenths, two for hundredths, three for thousandths, or more at pleasure; and with these proceed to operate as in the above example [861094 divided by 432] where I have added three ciphers; the quotient being 1993,273, signifies 1993 integers and 273 thousandth parts, or $\frac{273}{1000}$, or as Stevinus has it, 1993, 2^{I} 7^{II} 3^{III}." In the example referred to, the decimals are separated by a comma from the integers, but in the following passage (p. 6) taken from his "Logarithmorum Canonis Constructio," published in 1619, the decimals are separated by a period from the integers.

"The less accurate calculators take 100000 as the largest sine, the deeper select 10000000, by means of which number the difference betwixt all the sines can be better expressed. That is the reason why I have adopted it for the whole sine, and as the maximum of the geometrical progression. In computing tables, even very large numbers are to be made still larger by placing a period betwixt the original number, and ciphers added to it. Thus at the commencement of my computation I have changed 10000000 into 10000000·0000000, lest the most minute error might, by frequent multiplication, grow into an enormous one. In numbers so divided, whatever is noted after the period is a fraction, whose denominator is unity, with as many ciphers after it as there are figures after the period. Thus, 10000000·4 is equivalent to $10000000\frac{4}{10}$. So 25·803 is the same as $25\frac{803}{1000}$. Also 9999998·0005021 is $9999998\frac{5021}{10000000}$, and so on. From the tables so computed, the fractions placed after the period may be rejected without any sensible error, for in these very large numbers the error is to be considered insensible and nugatory where it does not exceed unity. For when the table is completed, for the numbers 9987643·8213051, which are equivalent to $9987643\frac{8213051}{10000000}$, there may be taken 9987643 without any sensible error."

Norton's tract did not reach a second edition, and the subject appears not to have been brought under general notice. About eleven years after, the substance of Norton's tract was published by Henry Lyte in 1619, with a dedication to Charles Prince of Wales.[1] His work contains no additions nor improvements on the notation of Stevin.

The notation adopted by Norton is somewhat in form different from that of Stevin. Instead of circles with small figures placed within them, Norton employed a parenthesis, with small figures placed between the upper ends; thus $8(^{0})9(^{1})3(^{2})7(^{4})$ means 8·9307.

In the first chapter of his Clavis, published in 1631, Oughtred explained the principle of decimals, and separated the integers from the decimals by the mark $\underline{|\quad}$, which he called the separatrix, as in p. 2 he writes $0\underline{|56}$, $0\underline{|00005}$, and $379\underline{|236}$, for ·56, ·00005, and 379·236 respectively. The theory as given by Oughtred and his notation were generally adopted by writers on arithmetic for more than thirty years after his time. Both the English and foreign writers on arithmetic adopted different modes of notation, all of them, however, following more or less the notation of sexagesimals.

The Arithmeticæ Theorea et Praxis of A. Tacquet, published in 1656, marks the places of decimals with Roman numerals as exponents, after the manner of Stevinus, who employed figures.

Briggs in the introduction to his Arithmetica Logarithmica employs a line placed under the decimals to distinguish them from the integral numbers; thus, p. 5, he writes, $3\underline{43}$ for 3·43, and $16\underline{807}$ for 16·807. In his posthumous work, "Trigonometria Britannica," published by Gellibrand, appears the same method of separating decimals and integers, as, p. 30, $1\underline{31595971}$ denotes 1·31595971.

It may be a matter for surprise that the convenience of Napier's simple notation for the separation of decimals from integers was so imperfectly appreciated at the time, that it was not generally adopted until after the middle of the seventeenth century.

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[1] The following is a copy of the title page: "The Art of Tens; or, Decimal Arithmetike; wherein the art of Arithmetike is taught in a more exact and perfect method, avoyding the intricacies of fractions. Exercised by Henry Lyte, gentleman, and by him set forth for his countries good. London, 1619."

Professor De Morgan in his "Arithmetical Books" (p. xxiii.) questions the fact of Napier having first applied the comma or period to separate integers from decimals. He remarks: "The inventor of the single decimal distinction, be it point or line, as in 123·456 or 123|456, is the person who first made this distinction a permanent language, not using it *merely as a rest in a process,* to be useful in pointing out afterwards how another process is to come on, or language is to be applied, but making it his final and permanent indication as well of the way of pointing out where the integers end and the fractions begin, as of the manner in which that distinction modifies operations. Now, first, I submit that Napier did not do this; secondly, that if he did do it, Richard Witt did it before him."

It is true that Richard Witt in 1613 published a work entitled "Arithmetical Questions touching the buying or exchange of annuities, &c., briefly resolved by means of certain breviats."[1] These are tables of compound interest calculated yearly, half-yearly, and quarterly, and a small vertical line is employed as a separation of the integers from the decimals. The tables are expressly said to consist of numerators with unity and ciphers annexed for denominators.

On the table of the amounts of £1 at compound interest at 10 per cent. per annum for one year to thirty years, he remarks,—"These 30 termes, viz., the 30 numbers [11, 121, 1331, &c.] in the table, are numerators of improper fractions. The denominators of which fractions are also a progression: the first term thereof (that is, the first denominator) being 10, the second ten times the first, which is 100, and the third ten times the second, which is 1000, and so on increasing to 30 terms. So it appeareth, that if the numbers (or numerators) in the tables be taken with their denominators they will stand thus, $\frac{11}{10}$, which is $1\frac{1}{10}$; $\frac{121}{100}$, which is $1\frac{21}{100}$; $\frac{1331}{1000}$, which is $1\frac{331}{1000}$; and so forth, till all the thirty termes have their denominators placed under them."

In p. 15 he writes the fraction $\frac{174494022}{10000000}$ thus, 17|4494022, and employs no other notation in his work. He used the period to separate pounds, shillings, and pence, as 6l. 13sh. 4d. or 6 . 13 . 4d.

It does not appear that Witt employed his separatrix as Napier had used the comma and period, in the full sense of its modern employment. This is clear from the examples in Napier's "Logarithmorum Canonis Constructio," which had been composed long before Witt's book was published.

Mr. De Morgan ingenuously adds, "But I can hardly admit him (Witt) to have arrived at the notation of the decimal point. For, though his tables are most distinctly stated to contain only numerators, the denominators of which are always unity followed by ciphers, and though he has arrived at a complete and permanent command of the decimal separator (which with him is a vertical line) in every operation, as is proved by many scores of instances, and though he never thinks of multiplying or dividing by a power of 10 in any other way than by altering the place of this decimal separator, yet I cannot see any reason to suppose that he gave a meaning to the quantity with its separator inserted. I apprehend that if asked what his 123|456 was, he would have answered:—It gives $123\frac{456}{1000}$, not it is $123\frac{456}{1000}$. It is a wire-drawn distinction; but what mathematician is there who does not know the great difference which so slight a change of idea has often led to? The person who first distinctly saw that the answer – 7 always implies that the problem requires seven things of the kind diametrically opposed to those which were assumed in the reasoning, made a great step in algebra. But some other stepped over his head, who first proposed to let – 7 *stand for* seven such diametrically opposite things."

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[1] In Jeake's Arithmetic, p. 427, "Practice is so called from the frequent use and general practice thereof, and is a compendium or breviat of the brief rules and most expeditious method of resolving the proportions resolvable by the rule of three." The word breviat is applied by Witt to his tables of interest.

William Oughtred, Etonensis (as he always styled himself), was born and educated at Eton College, whence in 1592 he was elected a scholar to King's College, Cambridge, and afterwards to a Fellowship. He devoted his attention chiefly to the mathematical sciences, and both by his example and his writings contributed to promote and extend the knowledge of them. His principal work was the "Clavis Mathematica," published in 1631. It passed through several editions, and was in repute for a considerable period; and it appears to have been the chief elementary work used in the Universities of Oxford and Cambridge. An English translation of the Key, "new forged and filed," was published in 1647, and dedicated by the author to Sir Richard Onslow and Arthur Onslow, Esq. In 1682 was published "Oughtredus Explicatus sive Commentarius in ejus Clavem Mathematicam, ad Juvenes Academicos, authore Gilberto Clark." This commentary it appears had been written by the author twenty years before, when he was a member of Sidney Sussex College, Cambridge. Another translation into English was afterwards made from the best edition, with notes, and published in 1694. Oughtred published several other works on the mathematics; and after his death, a selection from his papers was published in 1676 at Oxford, under the title of "Opuscula Mathematica hactenus inedita." Lilly, the astrologer, in his life styles William Oughtred the most famous mathematician then in Europe. After he had accepted the rectory of Aldbury, near Guildford, he continued his studies to the end of his life.

The late Dr. Peacock, Dean of Ely, remarks of him:—"In those days the members of these Royal Foundations had not yet begun to consider the pursuits of literature and science incompatible with each other. His works enjoyed a well-deserved reputation in his day, and he is spoken of in his old age with singular reverence by Wallis. He died in 1660, in his eighty-seventh year, from excess of joy on hearing of the restoration of the monarchy."

There are other names deserving of mention, both of Italy, France, Spain, Holland, and Germany, as well as of our own country, who have subsequently introduced improvements and promoted the extension of the knowledge of the Science of Number. The labour of performing calculations with large numbers has been considerably lessened by the extension of the denary scale to decimal fractions, and still more by the invention of Logarithms; the former extending the powers of the notation, and the latter perfecting the methods of computation of the Indian arithmetic.

In addition to the symbols assumed for the expression of numbers, other symbols have, from time to time, been devised to express the relations and the elementary operations of numbers. These symbols have been designated *symbols of operation*, to distinguish them from figures which have been named *symbols of number*. As to the symbols themselves, they stand simply as abbreviations of written words, and only require their assumed meaning to be understood so as to render intelligible the expressions in which they occur. The operations of arithmetic can dispense with the use of them; but where several operations are to be performed, some the reverse of others, it will be found useful to indicate the operations by means of these symbols.

The mark = was introduced and used by Robert Recorde for the sign of equality. The first account of its use occurs in his treatise on Algebra, entitled "The Whetstone of Witte." In the rule of equation he remarks:—"And to avoide the tediouse repetition of these woordes, *is equalle to,* I will set as I doe often in woorke use, a pair of paralleles, or gemowe lines of one length, thus: =, bicause noe 2 thynges can be more equalle."

When the symbol = is used in arithmetical reasonings or calculations, it must be understood as having relation only to pure arithmetical equality. Napier adopted it and defined it in these words:—"Betwixt the parts of an equation that are equal to each other, a double line is interposed, which is the sign of equation." Mr. Babbage, however, in one of his papers on Notation, observes:—"It is a curious circumstance that the symbol which now represents equality was first used to denote subtraction, in which sense it was applied by Albert Girard, and that a word signifying equality was always used instead until the time of Harriot."

The signs of relative magnitude, > meaning *is greater than,* and < *is less than,* when placed between any two numbers, were first introduced by Thomas Harriot, in his "Artis Analyticæ Praxis."

The sign + is used to signify addition, and was first employed by Michael Stifel in his "Arithmetica Integra," which was published in 1544. Its origin is uncertain; it has been supposed to be the abbreviated form of the word *et,* as found in some manuscripts. The mark + was used by Stifel for the word *plus,* and employed strictly as the arithmetical sign of addition, instead of the words "is added to."

The sign − was also employed by Stifel in the same work as the sign of subtraction. Some have supposed he adopted it from the fact that a small line — was commonly used in Latin writings to show the contraction of a word by the *omission* of one or more of its letters, as secūdū for secundum, numerūm for numerorum, &c., and that he named the mark *minus,* and used it instead of the words "taken from," or "subtracted from."

The sign × of multiplication was introduced by Oughtred in his "Clavis Mathematica." It is used to indicate the product of two numbers when placed between them, and stands for the words "multiplied into." The product of more numbers than two may be expressed in a similar manner.

The sign ÷ placed between two numbers denotes the division of the former by the latter, and stands for the words "divided by," or "is divided by." The Hindus placed the divisor under the dividend, with no line of separation. The line was afterwards introduced by the Arabians, and has since been universally adopted.

ELEMENTARY ARITHMETIC,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION II.

OF MONEY.

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MONEY.

> Above all things good policy is to be used, that the treasures and monies in a state be not gathered into few hands; for otherwise a state may have a good stock, and yet starve.—Bacon.

The money or coin of a State or Kingdom is the standard measure of the value of all things subject to valuation, and it is at the same time the equivalent in which all contracts are made payable. Money is therefore both a *measure* and an *equivalent*, and these two qualities can never be brought perfectly to agree. Money as a *measure* differs from all others; for if made of a material of small value, it would not answer the purpose of an *equivalent*. And if it be made, in order to answer the purpose of an *equivalent*, of a material of value subject to frequent variations, according to the price at which such material is sold in the market, it fails, on that account, in the quality of a *standard* or *measure*, and will not continue to be perfectly uniform and at all times the same. In civilized countries money has generally been made of gold, silver, and copper, but sometimes of other metals or of mixed metals. Certain portions of these metals, called coins, with an impression struck upon them by the order of the Sovereign power, as a guarantee of their purity and weight, serve as money. Coins made of gold or silver, or of any other metal, whether considered as a *measure*, or as an *equivalent*, are however subject to some imperfections. A failure or increase in the supply of one of the precious metals will tend to augment or diminish the value of the other. The supplies of gold, of late years, have been more largely augmented by the discovery of the gold fields in California and Australia, than at any previous period of the world's history, and gold has in consequence declined in value as compared with silver. It is evident, therefore, that coins made of gold and silver may vary in value with respect to each other at the same period in different countries, or in successive periods in the same country. As in England in the 43rd year of the reign of Queen Elizabeth, the relative value of fine gold and silver at the mint in that year was rated at less than 11 to 1; but in 1663 the rate had increased to above 14 to 1; and in 1805, it was rated at about 15 to 1. In the mints of foreign countries, the value of gold as compared with that of silver was estimated at a still higher rate.

As each of these metals may vary in its value with respect to the other, so each may vary in its value even with respect to itself; and this variation will be caused by the greater or less quantity that may happen to be in the market or in circulation. And further, if the supreme government of any country fix the rate or value at which coins made of different metals shall pass into currency, the relative value of the metals in the market may be found to differ from the value fixed in the coins: and if these coins of different metals are equally a legal tender, there will be two or more measures of property differing from each other. Besides the fluctuations in the relative value of gold and silver arising from demand and supply in commerce, there is another cause which has in past times very materially affected their relative values in this country.

Gold is said *to be lowered* in respect of silver, when a pound weight of the former *is declared by law* to be exchangeable for a less quantity of the latter than formerly; *and to be raised* when the contrary takes place. And silver is said to be lowered or raised when the pound weight of silver is exchanged for a greater or less weight of gold than before. There is another relation to be noted, when gold and silver remain in the same proportion to each other, but the value of both is altered with respect to some other commodities. This alteration may be effected, either by coining the pound weight of standard gold and silver into a greater number of pieces of the same assigned value, as before, or by causing the current coins to pass for more than their previous values. In either case, the value of both the gold and silver is enhanced, and one may be enhanced more than the other, when each is made the subject of comparison with other commodities. The coin which is to be the principal measure of property, ought to be made of one metal only according to Mr. Locke, who considers this money the measure of commerce and contracts; and observes, that "two metals such as gold and silver cannot be the measure of commerce both together in any country." The truth of this principle results from the nature and uses of money. But coins of one kind of metal do not conveniently answer all the purposes of money. Coins of gold are well adapted for large payments, but not for retail traffic; and coins of silver and copper would be too bulky for large payments, and therefore inconvenient. It is therefore necessary, that in commercial countries coins should be made of different metals. If gold in ingots or coins be taken as the standard or principal measure of property, silver and copper coins may be used as a legal tender only in a limited degree, so far only as they are authorized by the State. The coins of every State are the measure of property within that State according to the nominal value declared and authorized by the supreme power. If gold be the legal tender without limitation, the silver and copper moneys must be made subordinate and auxiliary, as supplying coins of small value into which gold cannot be divided without the inconvenience which arises from the smallness of the pieces.

The variations in the relative values of gold and silver are also affected by other considerations and causes which must ever have an influence on society and nations. A dread of public insecurity and political convulsions in a nation, will have a tendency to raise the value of the precious metals. War will tend to raise the value of silver more than the value of gold. A turn of commerce unfavourable to a country will also affect the relative values of gold and silver, and will tend to raise the price of that metal which is preferred by those nations with which the inhabitants have commercial dealings. And in large commercial concerns, gold will be preferred, as by means of it extensive exchanges can be carried on with the greatest facility. If the coinage in circulation in any country be defective in purity or weight of metal, all payments from that country in its exchanges with other countries, will be augmented in proportion to the defect of the coins from their real value. Hence, it follows, that the exchange is always against a country which has a faulty or defective coinage, and in favour of those countries whose coinage is abundant, pure, and in full weight.

Mr. Greaves more than two centuries ago, wrote the following judicious remark on that subject: "If those advantages which one country may make upon another in the mystery of exchanges and valuation of coins be not thoroughly discovered and prevented by such as sit at the helm of the State, it may fare with them after much commerce as with some bodies after much food, that instead of growing full and fat, they may pine away, and fall into irrecoverable consumption." [1]

In the early ages of the world, the exchange of one commodity for another is generally admitted to have been the primitive mode of traffic. In the course of time when such exchanges became inconvenient, necessity, the mother of invention, devised the use of money as a common measure of all commodities. Among the oldest records contained in the Book of Genesis no intimations are found of the use of money before the deluge. When, and by whom, the use of money was invented as a medium of exchange is unknown. There could be no want of it until mankind had multiplied and formed themselves into communities. Silver at a very early period of the world's history became a general representative of value, and was employed as a medium of traffic in the transactions recorded.

The first mention of money, a thousand pieces of silver, occurs in the twentieth chapter of Genesis. The second, in the twenty-third chapter where Abraham is stated to have weighed four hundred shekels of silver, "current money with the merchant," which he paid to Ephron for a place of burial. A third mention is that in the thirty-seventh chapter, where Joseph is sold into slavery for twenty pieces of silver by his brothers to some merchants who were trading with Egypt. And that money was measured by weight is again noticed in the forty-third chapter, when Jacob's sons carried money into Egypt to buy corn. All these instances sufficiently shew that the use of money existed at that early period of the world's history in Egypt and in the country which was afterwards called Palestine. And it is to be noted that though this money was sometimes called "pieces of silver," Abraham paid his four-hundred shekels by weight. The word shekel itself, from the Hebrew word to 'weigh,' implies as much; for money at the first seems to have been a merchandize exchanged for other commodities.

There are no traces of the existence of coined money to be found in any ancient Hebrew records. Their current money was generally in silver, and its relative value to gold is unknown in the early periods of their history. And it is now almost universally agreed that they had not any coined money before the time of the Maccabees.[2]

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[1] The little work entitled "Easy Lessons on Money Matters for the use of Young People," written by the late Archbishop Whately, is worthy of the attention of older people, as it contains excellent suggestions on some of the perplexing problems of the present time for adjusting equitably the claims of capital and labour.

[2] Morell, one of the best judges of ancient coins, allowed that all the Jewish shekels he had seen were coined after the time of the Maccabees. The Jews had no coined money till the reign of Antiochus Sidetes. Before that time, their money consisted of small laminæ or plates of silver, called (Gen. xlii., 27—35, Prov. vii., 20), δεσμοὶ and ἔνδεσμοι ἀργυρίου "bundles of money," such as might be tied together. Hence the δραχμὴ or half shekel was called *beka*, signifying

The standards of the Hebrew weights were preserved in their Sanctuary or Temple, a practice common in other nations. Hence by "the shekel of the Sanctuary" nothing more is meant than a just and exact weight (Exod. xxx. 13—15). It is evident that the first capitation tax for the service of the Tabernacle was half a shekel, and the shekel was a quarter of an ounce. But after the time of the Maccabees a new shekel of half an ounce was introduced, and still named *the shekel of the Sanctuary*. Whatever this was owing to, whether policy or necessity, it enabled the Jews to embellish and adorn their temple by doubling its revenues.

Herodotus (I. 94) expressly asserts that the Lydians were the earliest people who coined money of gold and silver, and employed it in traffic. In the neighbourhood of Sardis, the capital of Lydia, have been found both gold and silver coins, oblong or circular, but rude in character and device, indicating an early state of the art of coining, and possibly older than any known specimens of the coins of Greece. Herodotus had travelled in search of knowledge in Egypt, Assyria, Babylon, and Phœnicia, but he makes no mention of their coinage, and as yet no coins of these countries have been found as old as those of Lydia and of Greece.

The Parian Chronicle, however, assigns an earlier date to the origin of coined money, and gives the honour, not to the Lydians, but to the people of Ægina. This is reasserted by Ælian. Phidon, king of Argos is reported to have struck silver drachmæ in Ægina, 740, B.C. (Leake, *Num. Hellen.*). The gold Darics of Darius and succeeding kings of Persia are not improbably the next in point of antiquity to the coinage of the Lydians.

About 880 B.C. when Lycurgus reformed the Constitution of Sparta, the iron money adopted by that State was useless in any traffic with their neighbours, and from its nature could not have lasted long. In about half a century after, a dispute arose concerning money, and was the cause of the first Messenian War. It could scarcely have been the iron money of the Spartans that was in question.

Herodotus (III. 95) in reference to the revenues of Darius Hystaspes, states that gold was reckoned at that time to be thirteen times the value of silver. Herodotus read his History at Athens, B.C. 445. About 50 years after Herodotus, Plato in his Hipparchus asserts that the value of gold was twelve times that of silver.

Thebes was probably the only city in Greece which coined gold before Philip II, king of Macedon, who began his reign B.C. 358, and drew more than 1000 talents of gold from the mines near Mount Pangæus. With this gold he had a coin struck called a *Philippus*. (Diod. Sic. II., p. 88). About the same time the Phœnicians plundered the temple of Apollo at Delphi, and carried off gold to the amount of more than 10,000 talents. These circumstances rendered gold so plentiful, that, according to Menander, who was

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the plate, or shekel divided into two parts, as by its form might easily be done: and Festus has observed that the Roman didrachm was called sicilicus (siclus) for the same reason, quod semiunciam secet. (Hussey, *Essay on Ancient Weights*, p. 196).

The shekel of the Hebrews, the *σίκλος* of the Greeks, and the Sicilicus or siclus of the Romans were synonymous terms, and mean the same thing, a quarter of an ounce, differing very little in weight; the name and estimate of both coming most probably from the same original.

born B.C. 341, its value was estimated to be to that of silver as not more than ten to one.

The coins of the cities of Greece[1] and their colonies exhibit the finest specimens of the art of coining in earlier times. The coins of Syracuse reached the highest degree of perfection, and the remains of these are marvels of art. The Roman coinage was inferior to that of the Greeks, but from the age of Augustus to Hadrian, the Roman coins maintained a high degree of excellence in their execution.

The earliest coinage of the Romans was of copper in the time of Servius Tullius, and they are supposed to have borrowed the art from the Etruscans. The silver coinage of the *denarius* was not introduced before the year 266 B.C., and was so named as having the value of ten asses. The gold coinage according to Pliny (*Hist. Nat.* XXXIII. 13) was introduced 62 years after the introduction of the silver coinage. The largest gold coin was the *aureus*. Aurei and Semi-aurei were the only coins in gold for nearly 300 years. Until the time of Sulla the aureus continued at 30 silver denarii, but in the reign of Claudius it was reduced to 25.[2]

The coins of Rome during the Emperors form a most interesting series. The copper coins are of historical importance. The largest was the Sestercius, and from the age of Augustus was called nummus or æreus, of the value about twopence English. From the time of Augustus to Gallienus, the silver denarius contained 16 assaria.

The coins extant of people and nations preserve and indicate some of the authentic facts of history, and are of service in determining with accuracy the order of events, and the succession of kings and rulers. A few brief notices of the history and use of coined money in our own country from early times, may appear to be a subject neither devoid of interest nor unworthy of the attention of the student.

When Julius Cæsar first landed in Britain, about 55 B.C., he states in his Commentaries on the Gallic war, that the ancient Britons "use brass as gold money, or iron rings adjusted to a certain weight, instead of money," without further explanation.

Of the coins of the early British kings before the invasion of Julius Cæsar, very little is known. There is one described by Mr. Evans (Ancient British Coins), having on its reverse the letters

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[1] The valuable collection of Greek coins made by Colonel Leake was purchased for the University of Cambridge; and a catalogue of a selection of them, exhibited in the Fitzwilliam Museum, Cambridge, has been published by Churchill Babington, B.D., F.L.S., &c., Disney Professor of Archæology, pp. 51, 4to., 1867.

[2] It appears from the account of Pliny, that the value of gold to silver in the time of the republic was about 14½ to one; but this relative value did not long continue at Rome. Her armies had become victorious in every quarter of the world, and her conquests were rapid and extensive. She quickly became acquainted with the wealth of other nations and adopted such a policy as suited her own views of acquisition. About 189 B.C., the relative value of gold to silver was estimated at 10 to 1 (Liv. xxxviii., 11). At a later period, conquest and plunder appear to have been the objects of Cæsar's expeditions in Gaul. When he returned to Rome with the plunder he had seized in Gaul; the gold amassed, according to Suetonius (Jul. Cæs. c.lix), was so immense that a pound was sold for 750 denarii throughout Italy. If the *pound weight* be meant, the value of gold to silver was then as 9 is to 1 nearly, there being 84 denarii in the pound weight. If the *pound tale* be meant, in which 100 denarii were reckoned to the pound, the relative value would have been as 7½ to 1.

[3] Utuntur tamen ære ut nummo aureo, aut annulis ferreis ad certum pondus examinatis, pro nummis.—*Cæsar, De Bell. Gall. V. fol. Rom.* 1496, *Ed. Princeps.*

TASC, Tasciovanus, king of the Trinobantes, who probably reigned about B.C. 30 to A.D. 5. His son Cunobeline succeeded, and reigned until about 42, A.D. On his gold, silver, and brass coins are found CVN, CVNO, CVNOBELI, for Cunobelinus; and CAMV, CAMVL, for Camulodunum, Colchester, the capital of Cunobeline's kingdom. The Roman letters and other devices on his coins plainly indicate an imitation of the coins of Augustus. It is stated by Geoffrey of Monmouth that Cunobeline was brought up at Rome under Augustus, and that a friendship subsisted between this king and the Romans.[1]

A few years after the death of Cunobeline, Britain was subjugated a second time under Claudius more completely than before, and the edict issued, as related by Gildas (De Excid. Brit. c. 5), ordered that all current money should have the imperial stamp. This was the usual practice in all countries which the Romans reduced under their power. A triumph had been decreed to Claudius in 43 A.D. for his conquest of Britain, which was celebrated the next year, and a triumphal arch was also erected to him (Dio Cassius, II.). After this Claudius issued a gold coinage, of which specimens are extant. On the obverse of these coins is the head of Claudius laureated towards the right, with the circumscription TI. CLAVD. CAESAR. AVG. P. M. TR. P. VIIII. IMP. XVI.,[2] and on the reverse, the front of a triumphal arch, with a pediment inscribed DE BRITANN, and surmounted by an equestrian statue between two trophies. He issued also a silver coinage. On the obverse is the head of the Emperor towards the left, and the same superscription as on the gold coinage: on the reverse, DE BRITANN, with the figure of the Emperor in a quadriga, his right hand resting on the edge, and his left holding a sceptre surmounted with an eagle.

During the rule of the Emperor Hadrian 117 to 138 A.D., the Britons revolted, and the Caledonians destroyed some of the fortresses built by Agricola. Hadrian, on hearing of these tumults, hastened to Britain, and reduced the people to submission; and to protect the Northern frontier of the province, built the wall which extended from the river Tyne to the Eden. Of the coinages in brass during his reign, there is one of which the pieces are large, having on the obverse the head of Hadrian laureated to the right, with the superscription HADRIANVS. AVG. COS. III.; and on the reverse, a female figure seated, her right foot resting on a rock, her head resting on her right hand, and a spear in her left, by her side a large circular shield with the circumscription BRITANNIA. The usual letters S. C. are placed under the figure. The same figure is retained on the copper coinage of Great Britain at the present time.

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[1] See the Catalogue of a Selection from the British and English Coins in the Fitzwilliam Museum, Cambridge, by Churchill Babington, B.D., F.L.S., &c., Disney Professor of Archæology, pp. 14, 4to., 1867.

In the Fitzwilliam Museum at Cambridge is a coin of Cunobeline, presented by the Rev. W. Selwyn, D.D., the Lady Margaret Reader in Divinity. The coin is of gold and was found near Shepreth in 1869. On the obverse, which is concave, is the figure of a horse in the action of galloping with the head on the coin towards the right. At the back of the neck is a star of four rays, or some mark like it, and under the figure the letters CVNO. On the reverse is an ear of wheat across the coin. On the right side of it are the letters MV, with three pellets over them. On the left is an imperfect letter, the upper part like the letter A, the lower part of it worn away. It is not unlikely the letter C may not have been struck with the A, as the left half of the coin is not a complete half circle as the right.

[2] Tiberius Claudius Cæsar Augustus, Pontifex Maximus, Tribunitia Potestate nonum, Imperator decimum sextum.

Of the numerous brass coins of Antoninus Pius, 138 to 161 A.D., there is one having on the obverse his head to the right laureated, with the circumscription ANTONINVS. AVG. PIVS. P. P. TR. P. COS. III.; and on the reverse a female figure seated on a globe surrounded by waves, holding in her right hand a standard, in her left a javelin; her elbow resting on the edge of a large circular shield by her side, and beneath the figure, BRITAN. Another has the same on the obverse, but on the reverse is a female figure seated on a rock in an attitude of dejection; before her a large oval shield and a military standard with the superscription BRITANNIA. COS. IIII. This coin was struck in the fourth Consulate of Antoninus, and probably denotes that the campaign undertaken against the Brigantes was then ended.

Of the coins extant of Commodus, who ruled from 180 to 192 A.D., there are three with records of the war in Britain. These contain either the word BRITANNIA, or VICT. BRIT. or VIC. BRIT., Victoria Britannica, in allusion to his victories over the Caledonians who had passed the Roman wall and invaded the province, but were driven back by his general Uppius Marcellus.

In the reign of Severus, 193 to 211 A D, the Britons again revolted against their Roman masters. Herodian states that the governor of Britain wrote to Severus informing him of the rebellion, and requesting him to send reinforcements or to come in person at once to reduce the insurgents. Severus, with his usual rapidity, arrived in Britain with a large army, and, accompanied by his son Caracalla, advanced to meet the rebels, whom he worsted in several engagements, though not without great losses. The growing infirmities of Severus compelled him to leave his son to carry on the war, and he retired to York, where he died in the year 211 A.D. In his last moments he urged his generals to prosecute the war against the Caledonians until they were exterminated. "Omnia fui et nihil expedit" was the dying exclamation of this daring and successful despot, of whom scarcely one act of mercy or forbearance is recorded.

Both the brass and silver coins of Severus are of various types, and specimens of seven coinages at least, which are extant contain on their obverses VICTORIÆ BRITANNICÆ, VICTORIÆ BRIT., or VICT. BRIT., all containing allusion to his victories over the Britons.

Caracalla succeeded his father Severus and held the supreme power from 198 to 217 A.D. There are several types of gold, silver, and brass coins of this reign bearing similar devices and the same inscriptions respecting Britain as on those of his father. The same description applies to the coins of Geta, his brother, who succeeded him.

From the times of Caracalla and Geta to the reign of Diocletianus, no Roman coins are known to be extant bearing the name of Britain, and it is doubtful if any were minted in the province of Britain.

Carausius, a celebrated admiral, sailed over to Britain with the Roman Fleet, and usurped the imperial power, 287 A.D., and held it for six years. He arrived at a time when their discontent had rendered the Britons ripe for rebellion. Tacitus (Agric. XV.) writes, "that in his time the Britons groaned under the yoke of the Romans: they complained that instead of having one master, as formerly, they had then two; one was the Governor, who exercised his cruelty upon their persons, and revelled in their blood; the other was the procurator, who seized and confiscated their property."

There are numerous coins of Carausius, some of them bear the letters M.L., which are supposed to signify *Moneta Londinensis.* Others in copper, of the same usurper, bear C or CL for Colonia, or rather Camulodunum. The same letters appear on some of the coins of Allectus, his successor.

The brass coins of Constantine and of his two sons have on many of them the letters PLN or P . LON., *Pecunia Londinensis,* clearly shewing them to be the produce of the province of Britain. It is probable, that in the general reorganization of the Empire in 330, A.D. the mint of London (then the only one in Britain) was suppressed. On the later types (*e.g.* Constantinopolis), of Constantine and his family, the PLN and P . LON do not occur. But the London mint is now believed to have been revived by Magnus Maximus in 383, A.D., and the mint mark AUG. OB. occurring on some very rare gold solidi, (and also AUG. on an unique silver piece) is more probably Londinium Augusta, than Augusta Trevirorum (Trèves).[1] It appears that the Romans had ceased to commemorate on their coins their exploits in Britain after the reign of Caracalla.

The Romans maintained their power in Britain for nearly 400 years, and totally abandoned the island about the middle of the fifth century. It has been truly remarked that "from the first landing of Julius Cæsar to the final abandonment of the island by the Romans, the history of Britain presents, with few intervals, one long scene of cruelty and extortion. Barbarian retaliation frequently followed civilized aggression, and war and slaughter were often preferred by the wretched islanders to the grinding taxation of their oppressors." The Britons, finding themselves enfeebled and defenceless by the tyranny and oppression of their civilized masters, solicited the aid of the Saxons against the attacks of their neighbours. The Saxons came to their help as allies, but soon settled themselves, and gradually brought the whole country under their power. The Britons were driven into Wales, or became the subjects of the eight successive Anglo-Saxon Kings, as the invaders were able to establish their power in different parts of the Island. These eight kingdoms, called the Octarchy, bore the names of Kent, South Saxons, East Saxons, East Angles, West Saxons, Mercia, Deira and Bernicia. Little is known of the coinage of these different kingdoms besides the coins which are extant. The Saxon invaders most probably brought their own money, for the Saxon coins bear neither in form, type nor weight any resemblance to the Roman coins then current in the island. The kingdom of Kent was the first established. The most ancient coin known of the Saxons is the sceat, a silver coin, and found to weigh from 15 to 19 grains. The word sceat is purely Saxon, the same as *sceat,* a part, and may probably mean the smallest part of the shilling. A sceat of Ethelbert I., King of Kent, 561 to 616 A.D., is said to be the earliest Saxon coin which has been assigned. The word scilling or scylling occurs in the laws of this King.

Fines are reckoned by shillings, and by pennies, in the laws of Ina, who reigned over the West Saxons from 688 to 726 A.D.[2]

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[1] See De Salis in Numism. Chron. for 1867 (N. S. Vol. VII.) p. 61.

[2] The Anglo-Saxon version of the Gospels is sufficient to show the existence of such coins as half-pennies and farthings in use among the Anglo-Saxons at the time that version was made, not from the Greek or the Vulgate, but from the Vetus Italica, as the Rev. Dr. Bosworth has shewn satisfactorily in p. xi. of the preface

Of all the petty monarchies of the Octarchy, the kingdom of Mercia was by far the most wealthy, if we may form this conclusion from the number of its coins which have descended to modern times. The earliest specimen extant is of the reign of Edwald, who began his reign 716 A.D., nearly a century after its foundation as a separate kingdom. The coins of Offa, whose reign in Mercia was from 758 to 796 A.D., exhibit some of the most elegant specimens of the Anglo-Saxon coins. As he was at Rome in the Pontificate of Adrian I., it is probable he brought over Italian artists to whose skill the improvement of his coinage is to be imputed. The coins of his successor, Egbert, assumed the former rude appearance of the Saxon money.

At the beginning of the ninth century, Egbert, on the death of his brother Ethelbert,[1] succeeded to the throne of the West Saxons, and during his reign brought the whole of the petty kingdoms of the Octarchy under his dominion, and first gave the name of Anglia to his kingdom, of which he is considered the first sole monarch, though the States of the Octarchy were not completely united in one kingdom until the time of Edgar. The coins remaining of the West Saxons are inferior in their workmanship to all the Saxon moneys.

At the age of twenty-two, Alfred succeeded to the throne in 871, A.D. In despair for his country, or in distrust of himself, at first he hesitated, but at the unanimous desire of the people of Wessex, he accepted the regal power, and was crowned by the Archbishop of Canterbury. His reign was in constant disturbance by the incursions of the Danes, with whom he was engaged in contests almost to the time of his death. Notwithstanding his troubles, he surpassed his predecessors in literature and the arts, and was unwearied in his efforts to raise and advance the intelligence of his people. King Alfred believed "the Ten Commandments" to be the laws of God, and prefixed them to his laws, thus recognizing them as the only true foundation of the laws of a Christian state. His laws are silent on

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to his edition of the Gothic and Anglo-Saxon Gospels, with the versions of Wycliffe and Tyndale. The Gospels, as well as the remains of the Anglo-Saxon Pentateuch also, are conclusive with respect to shillings as a silver coin in use. In Matt. xxvi. 15, and xxvii. 3, 5, *τριάκοντα ἀργύρια* are rendered thrittig syllinga, thirty shillings, not thirty pieces of silver as in the authorised version. It appears also from Exod. xxi. 32, that thirty shekels of silver was the price of a slave in the Hebrew Law, the same that Judas got for the betrayal of his master. It is rendered in the LXX by *τριάκοντα ἀργύρια δίδραχμα*, and in the Anglo-Saxon version thrittig syllinga seolfres, thirty silver shillings. Their proverbial expression "*ne sceatt ne scylling*," signifying neither small nor great, was most probably formed in reference to their smallest and largest coins.

In the Anglo-Saxon laws of Athelstan, when a man forswore a debt, the form of words prescribed to be used, was " I owe him neither *sceatt ne scylling*," meaning, I owe him nothing, either little or much. And it is there also stated that 30,000 sceattæ are equal to 120 pounds weight.

A silver sceatta, preserved in the Fitzwilliam Museum, is described by Professor Churchill Babington in his catalogue of select British and English coins.

[1] A large hoard of the coins of Ethelbert, 249 in all, was discovered in 1817 in the neighbourhood of Dorking. In the year 1808 a large quantity of Saxon coins, (542 in number) was turned up by a plough in the parish of Kirk Oswald, in Cumberland. Some bear the names of the Kings Eanred, Ethelred, Redulf, Osbercht; and others the names of the Archbishops Eanbald, Vigmund, and Vulfhere.

the money and coinage of his time. His three mints[1] were at London, Canterbury, and Oxford. In the league between Alfred and Guthrum the Dane, both the mark[2] and the mancus are named. After the martyrdom of Edmund by the Danes, Guthrum was placed on the throne of East Anglia, and on his conversion to Christianity in 878 he took the name of Ethelstan. On the death of Alfred in 901, his son Edward, called the Elder, succeeded to the throne.

Athelstan, grandson of Alfred, began to reign 924 A.D., on the death of his father Edward the Elder. He was the first who ordained laws for the regulation of the coinage, and reduced the standard of the Saxon shilling from five pennies to four pennies. In a grand Synod four years after his succession, he decreed that there should be one kind of money in use throughout the kingdom, and that no one should coin but in a town. On his coins he is styled simply Rex, or Rex Saxonum, or Rex Totius Britanniæ, the last a title not found on any of the coins of his predecessors. He was the king of the West Saxons by inheritance, of Mercia by election, and of the greater part of the rest of Britain by conquest. In his laws occur the names of all the denominations of money in use among the Anglo-Saxons. He died in the year 940. A.D.

Edward II., called the Martyr, in 975 ascended the throne on the death of his father Edgar. He had mints both at Oxford and at Cambridge, and it does not appear that he made any alteration in his coins. He was put to death at the command of his stepmother to make way for her son Ethelred, who was placed on the throne in the year 979. The reign of Ethelred was marked by a series of actions which betrayed the most helpless irresolution and pusillanimity. The Danes invaded his kingdom, and levied exactions to the immense amount of 167,000 pounds. The last of these was made in 1014, and amounted to 30,000 pounds. His moneyers were numerous, and his laws shew that provision was made to preserve the integrity of his coinage. He fled to Normandy in 1013, and Swein, the Danish invader, mounted the throne. He died a few months after his elevation, and his son Canute, after a contest, was established in the kingdom in 1017. The mints of Canute were more numerous than those of his predecessors. He had one at Oxford and another at Cambridge, and ordained that one coin should be current throughout his kingdom, and that no man should refuse it, unless it were false. The standard both of the Danish and Anglo-Saxon money is somewhat uncertain. England under his reign became more wealthy, as being able to retain the fruits of her industry, which had been for more than two centuries before perpetually plundered by the rapacity of the Danes. At his death in 1036, his son Harold ruled in England. On Harold's death in 1040, Hardicanute, his brother, came to the crown. Upon his death, Edward, surnamed the Confessor,[1] the surviving son of Ethelred II., succeeded in 1042. On the death of his father he had been sent to Normandy, where he had lived till the time of his accession. His mints were numerous, and as many as 500 varieties of his coins are known, several of which indicate that they were struck at Cambridge or Oxford, as well as at many other places. The fines which were imposed by his laws were regulated by pounds, oras, marks, shillings and pennies.

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[1] In the first volume of the whole works of King Alfred the Great (2 vols., royal 8vo, 1858), will be found some account of the Anglo-Saxon Mint, with a description and impressions of all the coins of King Alfred known to be extant.

[2] The mark, a certain weight of money, gold or silver, was introduced into England most probably in the time of King Alfred, as marks of gold appear for the first time in the league between that monarch and Guthrum, about A.D. 878.

In the treaty between Edward and Guthrum, the fines to be imposed on Saxons and on Danes were stated differently; as for instance, the Saxon was to pay 30 scillings and the Dane a mark and a-half; and again in another part, where the Saxon was to forfeit 30 scillings, the Dane was fined 12 oras.

The silver mark was only in the tenth century estimated at 100 pennies, and at 160 in A.D. 1194 according to Matthew Paris. The value of the mark at 160 pence or 13*s.* 4*d.* was long continued in payments of legal fees, fines, &c.

In the Judicia Civitatis Londoniæ, before the middle of the tenth century, a schilling was reckoned at 12 pennies. About the end of that century, Archbishop Ælfric names the schilling at 5 pennies. This discrepancy seems to shew that there were two pennies in circulation, but of different values. And Wilkins has shewn (p. 66, 71) that two distinct estimates of the scilling occur by thrismas, which can only be reconciled by the supposition of the fact of two shillings of different values being current at the time. It is uncertain when arose the reckoning of 12 pennies to the solidus or shilling, and 20 shillings to the pound. That revenue was so reckoned in England in Saxon times, is clear from the Domesday Book:—"In Civitate Sciropescirie T. R. E. (*i.e.* Tempore Regis Edvardi) erant CCLII ..burgenses reddentes per annum VI lib. XVI solidos, et VIII dinar." In Athelstan's laws, the Thane's Weregild is computed at 1200 shillings, and said to be equal to the sixth part of 20 pounds. If 1200 shillings were equivalent to 20 pounds, the pound or 240 pence must have been equal to 60 shillings of 4 pennies each shilling. This division continued to the Norman times, and one of the Conqueror's laws places the matter beyond all doubt. It is there stated, that the Saxon shilling was 4 pennies, and the preamble declares that these laws were in force during Edward the Confessor's reign. "Ice les meismes, que le Reis Edward sun cosin tint devant lui del dei apres le polcier XV solz de solt Englois co est quer diners." The fine for the loss of the first finger was XV solz, *i.e.* of the English solt, or solidus, which was four pennies.

On the death of Edward the Confessor in 1066, there were four competitors for the crown of England, Harold, Earl Godwin, Tostig his brother, the king of Norway, and William, Duke of Normandy. Both Harold and William founded their titles to the crown on the gift of Edward.

Harold being in England assumed the sovereignty at once, and defended himself against his two brothers, Tostig and the king of Norway, and they were defeated and slain. His victory over them had so weakened his forces, that he was unable to withstand the attack of the Duke of Normandy, and fell at the battle of Hastings after he had held the supreme power for a little more than nine months. During his short reign he made frequent coinages at his numerous mints, one of which was at Oxford. The coins have on the obverse his name and title as king of England, and the reverse bears the moneyer's name and place of mintage.[2]

William, Duke of Normandy, 1066—1087, by the decisive battle of Hastings and the death of Harold became established on the throne of England. He did not rest his title by right of conquest only, but affected to consider himself the lawful heir and successor of Edward, whose laws he shortly after ratified. He used all his arts to cajole the people of England into the belief that he regarded them as his natural subjects, and not as a conquered people.[1] He imitated the form and type of the coinage of the late monarch Harold, and made no alteration of the standard used in the mints of the kingdom, two of which were at Cambridge and Oxford. In his charters he styled himself "Ego Willielmus Dei Gratia Anglorum hæreditario jure factus" (*Monast. Angl.* I., p. 44). He did not inscribe *Dei gratia* on his coins, which have on the obverse his name and title as king, with some of the leading letters of Anglorum. In the Domesday Book, wherever any notice of his arrival in England is referred to, it is always made (with one exception, fol. 124, *b*), as if he had come to his own dominions and without opposition.

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[1] The Anglo-Saxon kings conferred the privileges of coining on their subjects. King Edward the Confessor, on confirming the liberties of St. Edmund (St. Edmundsbury), gave to Abbot Baldwin a stamp and authority to have an exchange or mint, and to coin in his monastery. The coins now extant prove this, and shew that there were very few considerable towns without a mint.

[2] In the year 1739, a large quantity of Harold's coins was found at Dymchurch, in Romney Marsh; and again in 1774 there was another parcel found near St. Mary Hill Church, in London.

The Domesday Book furnished him with the exact amount of taxation he might impose in kind, or which might be commuted for money. It is recorded, that the sheriff was ordered to commute bread for 100 men or a pasture-fed ox for one shilling; a ram, sheep, or provender for 20 horses for fourpence. The Conqueror fixed the rate at which supplies or services might be commuted, and thus the value of the money was determined. He considered every man bound to pay according to what he had, and it does not appear he ever demanded what his subjects had not; like a prudent shepherd, he knew how much wool he could fleece his English flock; he was satisfied with the fleece, but seldom or never touched the carcase. And from the same document it appears he made little alteration in the laws and ancient customs which prevailed in cities and boroughs in the time of Edward. He retained the weight of the Saxon penny, but introduced the Norman mode of computation by shillings of 12 pennies, and the pound of 20 shillings. In his laws the fines are regulated by pounds, oras, marcs, shillings, and pennies. The shillings are sometimes stated expressly to be English shillings of four pennies each. In the Domesday Book various other coins or denominations of money are to be found; such as the mite, farthing, halfpenny, mark of gold and silver, ounce of gold and marsum. Of all these the penny, halfpenny, and farthing are the only coins which have descended to our times.

William II. (Rufus), 1087—1100, son of William I, succeeded his father on the throne of England. He soon dissipated the enormous wealth his father had accumulated. During his short reign he appears to have made no alteration in the standard of the coins.[1]

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[1] The conqueror entertaining an equal hatred towards the English and their language, determined to depress the one and annihilate the other. However superior their merit, the English were admitted to no dignity, but strangers were preferred. Their language was interdicted, and the laws of the English kings were translated into Norman French, and all his own laws were recorded in that language or in Latin. His efforts to impose the Norman French on his English subjects were unavailing, except to a very limited extent, as is clearly manifest from the predominance of the Saxon element in the English language of the present day. A statute of Edw. IV., so late as 1482, appears in Norman French.

When Gregory VII. (Hildebrand) despatched, in 1078, his legate to William, whose claims to the crown had been favoured by his predecessor, William replied—"Religious Father, your legate, Hubert, coming unto me, admonished me, in your behalf, inasmuch as I should do fealty to you and your successors. Fealty I would not do, nor will I; because I neither promised it, neither do I find that my predecessors ever did it to your predecessors." Such was the assertion of the Royal Supremacy of William, the first sovereign of the Norman line.

Henry I., 1100—1135, usurped the throne in the absence of Robert his elder brother, the lawful heir to the crown. He ordained many wise and politic laws for the benefit of his subjects, and resisted the encroaching policy of the court of Rome. He inflicted penalties for debasing the coinage, and in 1125 the severest sentence of the law was executed on 94 delinquents of the moneyers.

After this severity, he had recourse to a new coinage which became a new evil, as one of the old chroniclers writes, "by thus changing the money, all things became most dear, whereof a right sore famine ensued." The types of his coins are various, as also the inscriptions. On the obverse of some are his name and title as king of the English, on others, H.R., or Henricus. On the reverse are the names of the moneyer and the mint. When Henry died, he is reported to have left 100,000 pounds weight of silver in his treasury.

Stephen, 1135—1154, usurped the government, to the exclusion of Maude, the surviving daughter of Henry, and dissipated on his armies the immense treasures left by his predecessor. Henry, the son of Maude, Duke of Normandy, invaded England with a large army, and after a contest with Stephen, concluded a treaty which provided for his succession after the death of Stephen. His coins were reduced to a wretched state by his necessities and the powers his subjects had usurped. His pressing wants led him to sanction the debasement of the money of which he understood not the power.

Henry II., 1154—1189, succeeded as the lawful heir. After his coronation he took means to remedy the disorders which had arisen during the usurpation. He resumed the grants Stephen had made, and destroyed the castles illegally erected, whence had issued the greater part of the base coin which had brought great distress on the people. About the year 1156, he issued a new coinage,[2] as the money issued by his predecessor and by the numerous mints he allowed, was so debased in value that commerce was obstructed. He appreciated the power of money, and at his death was found in his coffers 900000 pounds weight of silver and gold, besides plate and jewels.

Richard I. 1189—1199, succeeded his father on the throne of England, and his martial disposition was unfavourable to the interests of his people, who were impoverished by the expenses of his armies, and the heavy ransom which was extorted for his release from unjust captivity. In the year before his death, his army defeated the French at Gisors. His parole for the day of the victory was *Dieu et mon droit*, which is still retained as the motto of the Royal Arms of England. It is uncertain how far he may have issued coins from English mints. There are specimens of the coins of his Gallic mints extant, which bear his name as king of England, though they were struck by him as Earl of Poitou or Duke of Aquitaine.

John, 1199—1216. John was born in 1166, and at a very early age, in 1177, had been declared by his father "Lord of Ireland," a title which seems to have invested him with regal power, as was manifest from his using a great seal, and striking money with his name and title impressed upon it. This title was most probably derived from a Bull of Pope Adrian IV., in which he gave permission to Henry II., in 1155, to subdue Ireland, and made use of the words "Et illius terræ populus te recipiat, et sicut Dominum veneretur" (Matthew Paris). On his accession to the crown of England, he had impressed the title of king on the coins he issued from the Irish mints, though on his great seal he continued the title of Dominus Hiberniæ. This title continued to be used by the English monarchs till the year 1541, when Henry VIII. assumed the title of king. His pennies and halfpennies have, on the obverse, his bust within a triangle, with Johannes or Johannes Rex only; on the reverse, a crescent and blazing star (perhaps intended for the sun and moon), besides a small star in each of the angles of the triangle. In order to discover coins deficient in weight, there was in his reign issued from his mint office a penny-poize, wanting one-eighth of a penny, to be delivered to any person who desired to have it, to be used until Easter in the next year.

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[1] In the year 1703—4, in digging the foundation for a new building in the Upper Ousegate, York, after a fire, which had happened there, a small oak box was discovered containing 250 pennies of William I and William II. And in 1833, at Beaworth, in Hampshire, a very large collection of coins of these reigns was discovered, amounting to about 6,500 pieces.

[2] A large quantity of the coins of Henry II. was found at Royston 1721, and subsequently a larger hoard of more than 5700 at Tealby, in Lincolnshire in 1807.

King John attempted to exact by force from his people the same heavy contributions as his father. His cruelty and oppression of the Jews almost exceed belief. He failed in his object, and was compelled to surrender the glory of his crown to the Pope,[1] and the power of it to the Barons, who, at Runnymede in 1215, wrested from king John the Magna Charta. The ratification of this charter has in successive reigns been confirmed at thirty different times.

Henry III., 1216—1272, was nine years old when his father died. On his accession, the greater part of his kingdom was in the possession of his enemies, and the royal treasury was exhausted.

In Grafton's Chronicle, it is recorded that in 1227, about this time a Parliament was holden in London, in which it was ordered, that "The English grote should be coyned at a certain weighte, and of the one side the King's picture, and on the other side a cross, as large fully as the grote to advoyd clippying." So greatly had the old coinage been lessened by clipping, that scarcely 20 shillings of the new coins could be gotten in exchange for 30 shillings of the old money.

The payment of tenths to the Pope of Rome was now exacted with so great severity, that people were compelled to borrow money of the usurers, who came over with Stephen, the Pope's Nuncio, at "the rate of one noble for the loane of 20 by the moneth," which was at the rate of 60 per cent. per annum. These usurers were banished about 1240, but being the Pope's money-changers, they were suffered to return in 1250, and were again expelled a short time afterwards.

1250. The king at this time extorted money from the Jews without mercy. Matthew Paris remarks, "that though he could make them wretched, he could not make them poor."

1257. The 41st year of Henry III. is remarkable for the first gold coinage[1] in England, of which any authentic records can be found, and it is extraordinary that it took place in the height of his distress for want of money. This event is related in a MS. Chronicle preserved in the archives of the City of London, which was probably written at the time, as the transactions are brought down to 1267. The writer states, that in this year the king made a penny of the finest gold, which weighed two sterlings, and willed that it should be current for 20 pence. On the obverse of the gold penny, the king is represented crowned, and sitting on a chair of state, with a sceptre in his right hand and a globe in his left. The reverse has the long cross of his later coinage, with a rose and three small pellets in each quarter. Scarcely had the king's proclamation been made, and the gold coins began to circulate, when the City of London made a representation against them, and obtained another proclamation that no one was obliged to take them, and whoever did, might bring them to the exchange and receive there the value at which they had been made current; one halfpenny only being deducted from each, probably for expense of coinage. In the 49th year of his reign, the gold penny was raised to 24 pence or two shillings.

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[1] In a subsequent reign, the Parliament unanimously declared, "that the grant of the kingdom by king John to the Pope was null and void; that it was made without the concurrence of Parliament, and in violation of his Coronation oath," and they determined that if the Pope should attempt, by process or otherwise, to maintain such usurpation, they would resist and withstand him with all their power.

From the Conquest till this reign, with the exception of the coins of Henry II. and John, great changes were made in the inscription and type of the coins. The portraits of the monarchs were represented either in profile or full, and the crosses were exhibited under almost every variety of form. The portrait of Henry III. is invariably full-faced on his coins, the cross consists of double lines; and the only difference between his earlier and later coinage is, that in the former the cross is bounded by the inner circle, and has four pellets in each quarter; while in the latter, it extends to the outer circle, and the number of the pellets is reduced to three. In this description the gold penny forms the only exception. This ornament, though rude, kept entire possession of the coins until Henry VII. introduced heraldic bearings. It then began gradually to give ground, but was not entirely lost before the time of James I., the end of a period of nearly 400 years.

Edward I., 1272—1307, succeeded to the turbulent reign of Henry III., whose ruling passions appear from his doings. Like his father, he was resolved to have money, and was not over scrupulous about the methods of obtaining it, from those who possessed it. His policy consisted in the skill with which he divided the wealthy classes against each other, and by attacking them singly, and bringing them by turns to bear upon each other, he so completely subjugated the whole as to render the power of the crown in his time as absolute as it had been in the time of the Conqueror. By means of extortion and confiscations he brought considerable wealth into his treasury. In his fifth year he had 300 Jews executed in London under the charge of debasing the coin, and their property he confiscated. And in his eighteenth year he banished 15,000 Jews from his kingdom, and confiscated all their goods. There was perhaps no class of his subjects, possessing money, from whom he did not continue to extort it. In his eighth year he issued a writ, which required all his subjects to shew by what titles they held their lands. And by this he drew large sums from the barons and others. One of the barons, however, John, Earl of Warren, responded to the king's demand, that he held his lands by the sword, and would so hold them as long as he lived until his death.

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[1] Both before and after the gold coinage of Henry III., Bezants (so named from Byzantium) and other foreign gold coins of various values were current in England.

Soon after his accession he applied himself, in 1274, to reform the abuses which had accumulated during his father's unquiet reign; one of them was the debasement of the money, which was carried to a height unknown before in the kingdom. The regulated produce of his mints was clipped[1] and otherwise reduced to less than half its legal weight, insomuch, that foreign merchants would not bring over their commodities, and every article of commerce became dearer.

No ordinances respecting the standard of the coins have been preserved between the Norman Conquest and the 8th year of Edward I. According to Stow (fol. 1633, p. 45), Gregorie Rokesley, Mayor of London, being chief master of the king's exchange or mints, a new coinage was ordered, that the pound of easterling money should contain 12 ounces, of which, fine silver, such as was then made into foil, and was commonly called silver of Guthuron's Lane,[2] 11 ounces, 2 easterlings, and 1 ferling, and the other 17 pence halfpenny to be alloy. Also the pound of money ought to weigh 20 shillings and 3 pence by account; so that no pound ought to be over 20 shillings and 4 pence; nor less than 20 shillings and 2 pence by account. The ounce to weigh 20 pence; the penny 24 grains, which 24 by weight then appointed, were as much as the former 32 grains of wheat.

It was ordained, that the money should be received and paid by the weight of five shillings in amount and five shillings in value by the tumbril, which was to be delivered by the warden of the exchange, being marked by the king's stamp as the measures were.

From the Conquest to 28 Edward I., the penny weighed 24 grains, Tower weight, or one pennyweight, so that a pound of silver was a pound both in weight and tale. But now the first variation from this rule took place, and the penny was reduced to 23·7073 grains Tower weight. This appears from an indenture in Lowndes' *Reports*, p. 34, which recites, that an indented trial piece of the goodness of old sterling was lodged in the exchequer, and every pound weight of such silver was to be shorn at 23 shillings and 3 pence.

In this year 1300, goldsmiths were restrained by statute from using any gold worse than the touch of Paris, or any silver inferior to the alloy of sterling. And no vessel of silver was to be delivered from the hands of the workman until it had been assayed by the wardens of the craft, and marked with a leopard's head (see Lowndes, p. 202). The coins of Edward I. and of his successors till Henry VII., represent him full-faced, and crowned with an open crown fleurie, consisting of three fleurs-de-lis, with two rays or lesser flowers, not rising so high as the other three placed between them. His style was, Rex Angliæ, Dominus Hiberniæ, which latter title he introduced upon English money, though it had appeared on the great seal from the time of king John.

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[1] The clipping of the coin was not confined to the laity or lower orders, but was practised by ecclesiastics of the highest ranks. Guy, the Prior of Montacute, was convicted of clipping, but pardoned by the King on payment of a fine of 60 marks in that year. He was a second time convicted of clipping and counterfeiting in the 13th year of the reign, and was again pardoned, but paid a fine of 200 marks, doubtless claiming the privilege of clergy (*Rymer's Foedera.* Ed. 1816, vol. i, p. 510. Clause Edw. I. m. 7).

The Chronicle of the Priory of Dunstable has recorded, that many Christians were accused by the Jews of having consented to the clipping of the coins, and these were chiefly of the nobility in London. In the month of July, in the same year, 1278, the King's Justices met at Bedford, to inquire who were the clippers of the money, and who had given consent and assistance to the Jews in that matter.

[2] So called from a former owner. It was a small lane leading out of Cheapside, east of Foster Lane, and was anciently inhabited by goldbeaters.—*Fuller's Worthies*, p. 201.

By 11 ounces, 2 easterlings, and 1 ferling must be understood, eleven ounces, two pennyweights and a quarter.—*Stow's Survey of London*, 1633, p. 45.

Edward II., 1307—1327, by proclamation, commanded in 1310, that money should be current at the value it bore in the reign of his father, and that no one should enhance the fine of his goods on that account, because it was the king's pleasure, that the coins should be kept up at the same value as they were wont to bear. It appears from the articles which the Commons delivered to the king, that the money was depreciated more than one-half.

Notwithstanding the measures taken, the base and clipped money continued to increase in the country. His coinage was in every respect exactly similar to that of his father. The favourable expectations formed on his accession, were disappointed; for the whole of his reign was turbulent and unfortunate. His frequent disputes with the barons left him little leisure for attention to restore the integrity of the coinage, or to frame statutes for the well-being of his kingdom.

Edward III., 1327—1377. The murder of Edward II. by Isabel his Queen and her confederates, placed her son, a youth, on the throne. This atrocious deed was done, under the idea, that his mother's influence over him, would place in her hands the government of the realm. In less than three years after his father's murder, she was imprisoned by her son, and condign punishment inflicted on her adherents. His next act was to disavow the excesses and abuses which had disgraced the beginning of his reign, as having been done without his authority, and he applied himself to correct them. Among these abuses, were the corruption of the coins of the realm, and the introduction of base money by foreigners. Other causes besides these contributed to the scarcity of money in the kingdom.[1]

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[1] At the Parliament held at Westminster, 28th May, 1343, a grievous complaint was exhibited by the Earls, Barons, Knights, Burgesses, and other of the Commons, for that strangers by virtue of reservations and provisions Apostolic, got the best of the benefices of the land into their hands, and never came to them, nor bore any charges due for the same; but diminishing the treasure of the realm, and conveying it forth, sore endangered the whole state. Thereupon a letter was framed by the Lords of the temporalty and Commons, representing the matter to the Pope, and signifying that they would not suffer such enormities any longer, and beseeching him to revoke such reservations, &c. This was most ungraciously received by His Holiness, who sent an answer; but the King, nevertheless, proceeded in prohibiting such provisions, &c., within his realm, on pain of imprisonment and death to the intruders.—*Holinshed* II. p. 365.

About half a century before this complaint of the English people was made, Pope Boniface VIII. in the year 1300, in imitation of the Jews (but not with the same object), first instituted a year of Jubilee, to be celebrated every hundredth year. The first celebration was found very profitable. Clement VI. reduced the period to 50 years. Urban VI. next ordained it to be held every 35 years, and lastly, Sixtus IV. further reduced the period to every 25th year. Erasmus has wittily remarked (Epp. Lib. xx. Ep. 90) "Monachi mirum in modum amant ignem purgatorium quod utilissimus sit illorum culinis. It is not unworthy of attention to note the various methods by which Papal Rome has successfully managed to subject the minds of men to her authority, and thereby to get command of their property.

In this reign an important change was made in the money of England, by the introduction of gold into the currency. Hitherto, the representative of value was limited chiefly, if not entirely, to silver, and the legal pound sterling by tale had been of equal value to the pound weight of silver. In the reign of Edward I., a small addition was made of three pence, so that a pound weight of silver was authorized to make one pound and three pence sterling. Such was the law until the proclamation for the gold coinage in 1343; and then it was in the power of the king to cause the pound sterling to be represented by a given weight of gold, as well as by a given weight of silver. Hence, it appears, that the gold coinage might at one time be made to conform to the standard value of silver, and at another time, the silver coinage might be made to conform to the standard value of gold.

Three sorts of gold money were ordered to be made in the Tower of London; that is, one coin with two leopards (lions), each piece current for 6*s*., and to be equal in weight to two petit florins of Florence, full weight; a second piece of one leopard, and a third of one helm, being the half and the quarter of the larger coin in weight and value. The type upon the largest coins bore an impression allusive to the royal arms of England, while the half bore a mantle on which the king's shield was displayed, quartering the arms of France and England, and the smallest piece was stamped with his crest. This money being rated too high according to the standard of silver, a new coinage of gold was made the same year. The Tower pound weight of gold of the old standard was coined into 39½ nobles, at 6*s*. 8*d*. each, and £13. 3*s*. 4*d*. by tale; or a proportional number of half and quarter nobles. On one side of this new coinage, is the king's image in a ship,[1] and on the reverse, a cross fleury with lioneux, inscribed "Jesus autem transiens per medium eorum ibat," and on the reverse of the quarter nobles, "Exaltabitur in gloria."

In 1347, the weight of the noble was diminished by nearly 10 grains, and the pound weight of gold was to make 42 nobles at 6*s*. 8*d*. each, and £14 by tale. At the same time the penny was reduced to 20 grains, or the pound of silver was shorn into 22*s*. 6*d*. by tale.

The year 1351 was remarkable for a great alteration which took place in the coins. They had hitherto been so much better than those of any other nation, that they were exported, and base money brought into the realm to the impoverishment of the people. New money of gold was made of like impression and value as before, but of less weight. The pound weight of gold, old standard, was coined into 45 nobles by tale, or a proportional number of half, and quarter nobles. Also a new silver coinage was issued. The pound weight of silver of the old sterling was to make by tale 75 grosses, or 150 half-grosses, or 300 sterlings going for a penny each, respectively amounting to 25 shillings. These were the first grosses coined in this reign. The

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[1] It is extraordinary, that the gold coins were not entitled from the new and singular type of a ship with which they were impressed, and thus distinguished from every other coin of that time existing. It could have been adopted only for commemorating some great event, most probably the victory of Edward over the French fleet off Sluys, on Midsummer day, 1340, when two French admirals and about 30,000 men were slain, and above 230 of their large ships were taken, with inconsiderable loss on the part of the English. (Carte, *Hist. of Eng.* Vol. II., p. 436).

words DEI GRATIA appear now for the first time on English coins. They were inscribed on his earliest gold coins, and afterwards on his groats, but not on his smaller pieces of silver. It is remarkable, that these emphatic words should not have obtained a place on the coins at an earlier period, as they are found on all the great seals after the reign of William I. They were used as early as the latter end of the seventh century by Ina, king of the West Saxons, in the introduction to his laws.

The 40th year of this reign is remarkable by an order of the king, that Peter-pence[1] should be no more gathered in England and paid to the Bishop of Rome.

Richard II., 1377—1399, at the age of twelve years ascended the throne of his grandfather. The troubles of his reign did not prevent attention to the state of the money and coinage. In the second year of his reign the Commons petitioned the king, against the clipping of the coins and the exportation of the good money. In the Parliament of 1381, the Commons again represented the distressed state of the country arising from the gold and silver being carried out of the kingdom; and what was left was very much clipped; they renewed their petition at the end of the Session, and prayed that good and wise counsel might be taken after inquiry and a remedy applied. The remedies proposed were various. One of the chief was, that no clerk nor provisor should be suffered to take away silver or gold, or to make any exchange for payments to the court of Rome. On these informations, a statute was enacted, which, among other things, provided that only "lords and other great men of the realm, and merchants and soldiers," might carry money in gold and silver out of the kingdom. A few years afterwards, an occasion arose for the enactment of a statute to protect the supremacy of the Crown, commonly called the statute of præmunire.[2]

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[1] In the year 1376, the court of Rome had become so oppressive in its exactions that the Commons were forced to prefer a petition to the Parliament in which they stated,—"So tost come le pape voet avoir monoie pur meintenir ses guerres de Lumbardie, ou aillours, pur despendre, ou pur raunson' ascuns de ses amys prisoners Frauncey pryses par Engleys, il voet avoir subside de clergie d' Engleterre. Et tontost ce luy est grantez par les prelatz, a cause que les evesqes n' osent luy contrestore, et est leve del clergie sanz lour assent eut avoir devant. Et les seculers seigneurs n'y preignent garde, ne ne font force coment le clergie est destruit, et la monoye du roialme malement emporte."

The King in his answer replied, that he had already provided against such offences by divers statutes and other means; and that he was then proceeding against the Holy Father, the Pope, in that very business, and that he was ready to do, from time to time, whatever might be effectual. It also appeared that the chief part of the money was conveyed out of the realm by the Pope's collector, and by the Cardinal's procurator, who likewise discovered the secrets of the state. The remedy proposed was, that Parliament should ordain, and that it should be proclaimed throughout the realm, that no one should in future, upon pain of life or limb, act as procurator for the Pope, the Cardinals, or any other residing at the Court of Rome.

In the following year, they further complain to the King "that the Pope's collector here held a receipt equal to a prince or a duke, and sent annually to Rome from the clergy for procuration of abbeys, priories, first-fruits, &c., twenty thousand marks, some years more, some less; and to cardinals and other clerks, holding benefices in England, was sent as much, besides what was conveyed to English clerks remaining there to solicit the affairs of the nation; upon which they desire his majesty, that no collector of the Pope may reside in England."

[2] The Commons of the Realm made the following noble declaration against the assumptions and aggressions of the Bishop of Rome:—

"The Crown of England hath been so free at all times, that it hath been

The coinage of Richard II. does not differ from that of his predecessors. His name is inscribed RICARD. or RICARDUS, with the title REX ANGLIE ET FRANCIE ET DOMINUS HIBERNIE et AQUITANIE. The last appears only on his gold coins.

Henry IV., 1399—1413. The system of favouritism of the irresolute Richard at length brought on his deposition, and his death by violence. He had been compelled to resign his crown from his incapacity to govern, and Henry, Duke of Lancaster, claimed the crown by hereditary right.

In the Parliament at Westminster, 20 Jan. 1400, the Commons petitioned the king, and the king's answer is contained in the statute of 5 Rich. II., which provided that gold and silver should not be sent out of the kingdom without the king's licence.

In the year 1402, the Commons complained that the statute 14 Rich. II., c. 2, respecting the exchanges made by merchants to the court of Rome had not been enforced.

In 1404, the statute 6 Hen. IV. c. 1, was intended to prevent the payment to the Church of Rome of more for first-fruits than had been accustomed, by which large sums had been carried out of the realm. Another statute was made in this year to prevent the increasing evil of carrying money out of the realm to the Court of Rome.

In 1411, it was ordained that the Tower pound of gold should be coined into 50 nobles, and the pound of silver into 30 shillings of sterlings, so that this gold and silver should be of the same standard as the old money. This ordinance brought down the coins, the groat to 60 grains, the half-groat to 30, the penny to 15, the half-penny to 7½, and the farthing to 3¾ grains. It will be seen, that the

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in no earthly subjection, but immediately subject to God in all things touching the regality of the same crown, and to none other, should be submitted to the Pope, and the laws and statutes of the realm by him defeated, and avoided at his will, in perpetual destruction of the King our Lord, his crown, his regality, and of all his realm, which God defend. And, moreover, the Commons aforesaid say, that the said things so attempted by the Pope be clearly against the King's crown and his regality, used and approved of the time of all his progenitors; wherefore they and all the liege Commons of the same realm, will stand with our said Lord the King, and his said crown, and his regality, in the cases aforesaid: and in all other cases attempted against him, his crown, and his regality in all points, to live and to die." The occasion of this declaration arose from the fact, that the bishop of Rome had ordained and proposed to translate some prelates of the realm, some out of the realm, and some from one bishoprick into another, within the realm, without the King's assent and knowledge, and without the assent of the prelates so to be translated. By which translations (if they should be suffered) the statutes of the realm would be defeated and made void; and the king's liege sages of his Council, without his assent, and against his will, carried away and gotten out of his realm, and the substance and treasure of the realm carried away, and so the kingdom would be destitute as well of counsel as of substance. To remedy these evils, in the year 1392, the statute 16 Rich. II. c. 5 was enacted. This statute declared, that all persons who should purchase or pursue, or cause to be purchased or pursued, in the court of Rome, or elsewhere, any such translations, should be put out of the King's protection; their lands and tenements, goods and chattels, forfeited to the King, and their bodies to be attached, if they might be found, and brought before the King and Council, there to answer; or that process should be made against them by *præmunire facias*.

In 1399, in order to check the carrying of money and gold and silver out of the realm by the Pope's agents, an oath was now administered by the collector, by which he engaged not to convey any money, &c., beyond the sea, without the King's special licence.

precious metals were not only enhanced, but their relative value was again altered.

Henry V., 1413—1422. The English coins of this monarch, both of gold and silver, were of the same weight as those of his predecessor. No means are known by which they may be distinguished.

Soon after his accession he directed his attention to the coinage of his kingdom, and finding that the treasure of his realm was fraudulently exported by alien Frenchmen, who had been appointed to benefices within the kingdom, contrary to the statute of 13 Rich. II., it was enacted in his first Parliament that the ordinances against such practices should be firmly holden and kept, and duly put into execution: and the alterations in the standard of coins by statute 13 Hen. IV. were established.

In 1414 no Parliaments were holden. The Commons represented the mischiefs which would ensue, whenever peace was made with France, from the sums of money which would year by year be remitted from the alien priories in England, to their chief houses abroad; and petitioned the king to take such priories into the hands of himself and his heirs for ever, with some few exceptions which they stated in their petition. To this the king gave his consent.

1420. In this year the Commons petitioned the king—that no person of what estate or condition soever should convey or carry out of the realm of England, or cause to be conveyed or carried, gold or silver for traffic, or for benefice of holy church, or for any other grace or privilege of holy church, or for any other cause whatever in the courts of holy church beyond sea, reasonable costs and expenses for their passage excepted, upon pain of forfeiture of life and limb, and of lands and tenements in fee simple, goods and chattels as in case of felony. It was not, however, thought fit to grant this petition, nor to go the length of making this offence capital. Accordingly, it was only commanded that the statutes already made should be observed and kept.

Henry VI., 1422—1461, was eight months old when his father died, and about two months after he was also proclaimed king of France on the death of Charles VI.

In 1432, the Pope's ambassador had licence to pass out of the kingdom with gold, money, and jewels, to the amount of one hundred pounds. The next year, the laws which prohibited the exportation of money were again partially suspended, and the bishop of Winchester had licence to carry out of the realm money and plate to the amount of 20,000 pounds of sterlings.

In this reign the usurpations of the Church of Rome, and the attempt of the Pope to render null and void the statute of præmunire, occasioned much trouble in the kingdom. The Commons presented an address to the king, praying that he would send an ambassador forthwith to the Pope to justify the conduct of Chicheley, Abp. of Canterbury, in refusing to consecrate a nominee of the Pope for a diocese in England. In the letter sent are the following memorable words: "Be it known to your Holiness, that while I live, by God's assistance, the authorities and usages of the Kingdom of England, shall never be diminished. But even if I were willing so to debase myself (which God forbid), my nobles, and the whole people of England, will by no means suffer it."

In the year 1444, on account of the want of small money, which had occasioned much distress among poor people, the Commons petitioned the Parliament for relief. Their petition concludes with these words: "This for the love of God, and for the common profit of the poure liege peple, which for this meritory dede shall hertly pray to God for you." The petition was granted. And some years after, he ordered the Tower pound of gold of old standard to be coined into £22. 10*s.* by tale, that is, to make 67½ angels at 6*s.* 8*d.* each, and the pound weight of silver of the old sterling into 112½ groats, amounting to 37*s.* 6*d.* by tale, and the half-groats, pennies, half-pennies, and farthings in proportion.

Edward IV., 1461—1483. In the fourth year of his reign, a new coinage was issued. The gold coins were then reduced to £20. 16*s.* 8*d.* by tale, that is, each pound of gold was to make 50 nobles, value by tale 8*s.* 4*d.* each. The silver was reduced to 37*s.* 6*d.* the pound weight.

In 1465 the gold coins were again altered, 45 nobles were to be made out of the pound of gold, each 10*s.*, or 67½ of the pieces impressed with angels, current for 6*s.* 8*d.* each, and consequently the pound weight of gold was coined into £22. 10*s.* by tale.

The silver moneys were the same as in the preceding year, at 37*s.* 6*d.* the Tower pound, so that there were made of groats 112½, half-groats 225, sterlings 450, half-pennies 900, and farthings 1800, all of the old standard. These new nobles were called *rials,* a term from the French, who gave that name to their coin, on account of the figure of the king in his royal robes (Le Blanc), but which was ill-applied to coins bearing the same impression as former nobles. The change of the name was probably intended to obviate the inconvenience which might have resulted from the nobles in currency, and the nobles in account being of different values. The new species of money, called the angel, being of the value of the noble, was called the noble angel.

On the reverse of the noble was the figure of the sun, then introduced on the coins of Edward IV., surmounted by a rose, the badge of the House of York. This impress he adopted after his victory at the battle of Mortimer's Cross, in Herefordshire, where, just before the battle, the extraordinary phenomenon of three suns appeared, which shone for a time, and then were suddenly united into one.

The inscription on the obverse of his English silver coins was "EDWARDUS DEI GRATIA REX ANGLIE ET FRANCIE," and on the reverse, "Posui Deum Adjutorem meum." On some it was Edwardus Dei Gratia Dominus Hibernie; on others, Edwardus Rex Anglie Francie, is continued on the reverse, Dominus Hibernie. On his great seal the style, which had been discontinued by Henry VI. he resumed in this form: Edwardus, Dei Gratia Rex Anglie et Francie et Dominus Hibernie. He made no alteration in the type of his coins, which are distinguished from those of his predecessors only by the name, weight, or mint-marks. He was the first English king who used a flaming sun as the royal badge on his coins.

1477. About his 17th year, the coins and bullion of the realm appear to have been debased by almost every possible method; for in a statute made in that year, the principal acts formerly passed for their preservation were recited, and fresh provisions enacted for the protection of them, which the infringement of the laws rendered absolutely necessary at that time. The provisions for remedy by this new enactment were to continue for seven years.

Edward V. was about twelve years old at the time of his father's death in 1483, and was murdered the same year.

Richard III., 1483—1485, the uncle of Edward V. usurped the throne, and one Parliament was held during his short reign. One of its provisions was a statute designed to stop the exportation of the coins, and another to remedy the abuses which had increased in the Irish mints during the last three years of Edward IV. The same type and legend appear on his coins as on those of Edward IV.

Henry VII., 1485—1509. The battle of Bosworth Field placed Henry, Duke of Richmond, on the throne, with the title of Henry VII. In the year 1489, the fifth year of his reign, a new gold coin was ordered of the old standard of the realm, but double of the value of the royal (rial). The Tower pound weight of gold was to be made into coins 22½ pieces by tale, each to be called a sovereign, and to be of value in payment for 20 shillings sterlings. The statute of the 19th year of his reign mentions half-sovereigns likewise. This statute also ordains, "that clipped money shall not be current in payments. And to prevent the clipping of coin in future, it is directed that a circle shall be about the outer part of the new coins, and that the whole scripture shall be about every piece, without lacking of any part thereof, to the intent, that the king's subjects hereafter may have perfect knowledge by that circle or scripture when the same coins be clipped or impaired." Although this monarch made no alteration in the standard of the metal, he introduced several variations from the usual type. He first placed on the coins an arched crown with a globe and a cross on the arch. The type of his coins in the 19th year was wholly changed. His portrait was given in profile, with a crown of one arch only, a form which had not appeared on the coins since the reign of King Stephen. A single beaded line likewise took place of the double tressure upon the obverse of the groats and half-groats; the inner circle of the reverse, which contained the name of the mint, was omitted; and the rude pellets, which had so long occupied the quarters of the cross, were superseded by an escutcheon of the royal arms surmounted by the cross. On some of these coins he added to his name Septimus or VII., a practice which had been disused ever since the reign of Henry III., on whose coins alone, of all our monarchs, from the earliest times, numerals or any other distinction of the kind had disappeared. The omission of I, II, III, upon the coins of the first three Edwards, and of IV, V, VI, on three of the Henrys, has occasioned difficulties almost insuperable in the appropriation of their respective coins to those monarchs.

The type of the usual gold money was continued nearly the same as before; but Henry's new coin, the sovereign, bore on the obverse the monarch seated in state upon his throne, from whence it derived its name, and on the reverse a double rose, in allusion to the union of the two Houses of Lancaster and York, with the royal arms in the centre.

On his silver coins he is styled HENRICUS DEI GRATIA REX ANGLIE ET FRANCIE; to which is added on his gold coins, DOMINUS HIBERNIE, as on those of his predecessors. On some of his ecclesiastical pennies, the king is represented on the throne, crowned and in royal robes, in his right hand a sceptre, and in his left a globe.

Henry VIII., 1509—1547. When this monarch came to the throne, he succeeded to the immense wealth the avarice of his father had accumulated. According to Lord Bacon (*Hist.* Hen. VII. p. 230) he left £1,800,000 sterling. And Sir R. Cotton writes, that he left in bullion four and a-half millions, besides his plate, jewels, and rich attire. All this wealth Henry dissipated in a few years, and to supply his extravagance for the future, he had recourse to disgraceful means, and stands recorded as the first of the English Sovereigns who debased the sterling fineness of the coins, and then legalized the debased coins.

On Sept. 23, 1513, King Henry, with his army, took Tournay, in Flanders, and in the same year had small silver coins struck in that city. They were groats of three kinds. One has his arms on the obverse, with the figure 8 after his name, and Civitas Tornacensis with the date 1513 on the reverse. This is the earliest date known in Arabic numerals on the coins of any English monarch.

On July 24, 1526, a writ was issued to Thomas Wolsey, Cardinal Abp. of York, legate a latere of the Apostolic See, Primate of England and Chancellor, commanding him to carry out into effect the king's design of reducing the English money to the standard of foreign coins. The reason assigned in a proclamation of August 22 following, was that the price of gold in Flanders and in France was rated so high as to cause all the coins of the realm to be transported thither. It was ordained by proclamation on 5 November, that the money of gold and silver should be reduced in weight.

One pound Troy of gold of the *old standard* was ordained to be £27 by tale, and to be coined into 24 sovereigns at 22*s.* 6*d.* each; and rials, angels, George nobles, half-angels, and penny pieces in proportion.

And one pound Troy of gold of 22 carats fine only was to be coined into 100½ double rose crowns; or 201 half-crowns, making by tale £25. 2*s.* 6*d.*

One pound Troy of silver of old sterling was to be coined into 135 groats or 275 half-groats, or a proportional number of sterlings, half-pennies, or farthings, so that every pound weight made 45*s.* by tale. The augmentation of the value of money by royal authority brought on the natural consequences, and loud complaints were made by the people.[1]

During the preceding reigns, all the provisions enacted to keep the good English coinage in the kingdom appear to have failed. Sir Thomas Gresham urged on Sir Thomas Audley, the Lord Privy Seal (25 July, 1529), the necessity of permitting all merchants, both subjects and foreigners, to exercise exchanges and rechanges without restraint, the want of which was a great detriment to trade, and requested him to prevail with his Majesty to issue his royal proclamation for that purpose, which was done when Sir T. Audley was Lord Chancellor.

The debasement of the standard of gold and silver was preceded by a proclamation in rather ambiguous terms, the year before the Act of Parliament confirmed it. By the indenture then made, 1543, a pound of gold of 23 carats fine and one alloy was to be coined into £28. 16*s.* by tale. The sovereigns were to be current at 20 shillings each, and the other coins in proportion. A pound of silver of 10 ounces fine and 2 alloy was to be coined into 48 shillings by tale; namely, into *testoons*,[1] 12 pence each, groats, half-groats, halfpence and farthings.

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[1] In Rastell's Chronicle, 1529, may be read one of them: "But ye must note that XLs. in those days [in the time of Richard II.] was better than XLs. of the present day, which is now the XXI yeare of Kynge Henry the VIII, for at those days V grotes made an ounce, and now at this day XI grotes maketh an ounce."—*London, reprinted* 1811, 4to. p. 242.

An Act of Parliament, which was passed the same year, ratified the style he had assumed, and declared it in the words: "Henricus Octavus Dei Gratia, Angliæ, Franciæ et Hiberniæ Rex, Fidei Defensor, et in terra Ecclesiæ Anglicanæ et Hiberniæ Supremum Caput:" and in English, "Henry the Eighth, by the grace of God, King of England, France and Ireland, Defender of the Faith, and of the Church of England and also of Ireland, on earth the Supreme Head." It was also declared to be high treason to attempt to deprive the king of this style (Stat. 35 Hen. VIII. c. 3).[2]

By a proclamation in 1544, the price of gold of 24 carats was fixed at 48*s.* the ounce, and the finest sterling silver at 4*s.* the ounce; and ordained the rate at which certain coins should be current. In this year the standard was still further reduced both in weight and fineness. The gold was to be 22 carats fine and 2 carats alloy, and one pound weight of gold was to be coined into £30 by tale, that is, 30 sovereigns of 20 shillings or 60 half-sovereigns; and the king had a royalty of 2 carats of fine gold for coinage, which yielded him 50*s.* on every pound weight of gold. The silver was reduced to 6 ounces fine and 6 ounces alloy, and the pound weight was coined into 48*s.* by tale, in testoons, groats, half-groats, pence, halfpence, and farthings.

In the year 1545, the coins were reduced to the lowest degree of fineness which ever disgraced the English mint, excepting a small quantity of silver in the first year of Edward VI. The gold was now brought down to 20 carats fine and 4 carats alloy; and the silver to 4 ounces fine and 8 ounces alloy. The coins continued to be of the same weight as they were in the indenture of the preceding year, but the debasement raised the pound weight of fine gold to £36, and of fine silver to £7. 4*s.*

The proceedings of Henry VIII. in the debasement of the silver coinage, first by making a fifth part alloy in the 34th year of his reign, and two years after making it half copper, and in his 37th year making it only one-third silver, and in like manner debasing his gold coinage, at length produced consequences which were suffered in the reigns of the sovereigns who succeeded him.[1]

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[1] Le Blanc says, the new species of coins struck by Louis XII. were called *testons*, because the head (teste, tête) of the monarch was represented upon them. The reason of adopting this name for the English coin is not obvious.

[2] In the Statute 23, Hen. VIII. passed in 1532, for the restraint of payment of first fruits to the See of Rome, it was stated that since the second year of Henry VII to this time, the sum of 80,000 ducats, amounting in sterling money to £160,000 at the least, had been paid for the investiture of bishops; and other great and intolerable sums yearly had been conveyed to the Court of Rome to the great impoverishment of the Realm. On this account it was ordained, that such unlawful payment of first fruits should utterly cease, and also the conveyance of sums of money to the Court of Rome under other pretences. This was further enforced by Statute 25, Hen. VIII, so that the payment of Peterpence and all other payments to the See of Rome were absolutely forbidden.

The author of "the Wealth of Nations," who was no bigot nor enthusiast, has recorded his deliberate judgment of the Church of Rome with its head, generals and orders of the Papal army,—"that it is the most formidable combination that ever was formed against the authority and security of civil government, as well as against the liberty, reason, and happiness of mankind."

It is a fact worthy of remark, that although Henry VIII. managed to enhance the silver coinage of England largely above its commercial value, the gold coinage was never so enhanced. He no doubt intended to make a like profit by the debasement of the gold coinage as he had succeeded in making of the silver coinage, but this profit could only be gained by making one metal dearer than the other. He had already raised silver to about four times its former value, while gold was raised only about one-third, and thus the relation as coinage was greatly altered; while the commercial value was not so altered. The gold was therefore kept nearly at its commercial value, and could be employed to purchase silver at its commercial value; and as silver had been raised by the king's proclamation to bear so exorbitant a value in coinage above its commercial value, it was possible for the merchant to secure considerable profits so long as the stock of gold in the country was not exhausted.

Edward VI. 1547—1553, was a minor on his accession, and the Regency added to the debasement of the coinage. The baseness of the testoons coined in the first year of his reign gave occasion to great complaints, and to several epigrams which were circulated in manuscript at the time.[2]

In 1549, the testoons were called in by proclamation, and by an indenture of the same year a new issue of coins was somewhat improved in fineness though reduced in weight. The pound of gold of 22 carats fine and 2 alloy was to be coined into £34 by tale into sovereigns of 20 shillings, and crowns of 5 shillings a piece, with halves of proportional value. The pound of silver of 6 ounces fine and 6 ounces alloy into 72 shillings of 12 pence each by tale, of which the merchant, for every pound weight of fine silver, received £3. 4*s.*, and the Crown about £4 gain (Lowndes, p. 46). These testoons or shillings, thus reduced in weight, but a little improved in fineness, are those most probably which Bishop Latimer alluded to in his two sermons before the king on 8 March and on 22 March, 1549, which gave considerable offence.[3]

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[1] Rents of lands and tenements, with the prices of victuals, were raised far beyond the former rates, and, as Strype in his time remarks, had hardly been brought down since. It is probable that Erasmus in his Adagia, p. 130, in speaking of the *Plumbeos Angliæ*, might refer to the debased silver coins. The inspection of one of them, a groat of Henry VIII. in the Fitzwilliam Museum at Cambridge, will fully warrant the justice of the expression. It is suggestive, that some of the base coins of Henry VIII. have the following words on the reverse, "Redde cuique quod suum est."

It appears from a passage of Budelius de Monetis, p. 5, that the leaden tokens mentioned by Erasmus as current in the reign of Henry VIII, still continued in circulation so late as the year 1591, the 33 of Elizabeth.

[2] The following are two of the epigrams alluded to, copied from John Heywood's Woorkes, printed in London, 1566.

63.
Of Testons.
Testons be gone to Oxforde, God be their speede:
To studie in Breasennose there to proceede.

64.
Of Redde Testons.
These Testons look redde: how like you the same?
'Tis a tooken of grace: they blush for shame.

[3] The following passage alludes to the new coinage: "We have now a prety

By proclamation on 30 April, 1551, the debased groats and shillings of Henry VIII. were ordered to be current for 3*d.* and 9*d.* on pain of forfeiture and imprisonment, with a further fine at his Majesty's pleasure. By another proclamation of 11 May was declared the king's determination to proceed in the restoration of the fineness of the coinage. On 18 July, a third proclamation was issued to quiet the alarm raised by the reduction in the value of coins, threatening severe penalties against all who should "invent, speak, mention, or devise any manner of tale, news or report, either touching the abasing of the said coins, or that in any manner of wise might sound either to the dishonour of his Majesty's person, or the defacing of his Highness's [the Regent, Duke of Somerset] proceedings, or of his council, or to the disquieting of his subjects, on pain of six months' imprisonment and such fine, &c." Notwithstanding the severity of this proclamation, it appears from the king's own journal, that in less

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little shillyng, indeed a very prety one. I have but one I thynke in my purse, and the last day I had put it away almost for an old grote, and so I trust some will take them. The finesse of the silver I cannot see. But therein is printed a fine sentence; that is, *Timor Domini fons vitæ vel sapientiæ*,—The feare of the Lord is the fountayne of lyfe or wisedome. I would God this sentence were alwayes printed in the hart of the king, in chosing hys wyfe, and in all hys officers."

In the Sermon preached on March 22, he points out the pernicious consequences of debasing the coins of the kingdom:—"Thus they burdened me ever with sedition, so this gentleman commeth up now with sedition. And wot ye what? I chaunced in my last sermon to speake a mery word of the new shillyng (to refreshe my auditorie), how I was lyke to put away my new shillyng for an olde groat, I was herein noted to speake seditiously. Yet I comforte myselfe in one thinge that I am not alone, and that I have a fellow . . . I have now got one fellow more, a companion of sedition, and wot ye who is my fellow? Esay the prophet. I spake but of a little preaty shillyng; but he speaketh to Hierusalem after another sort, and was so bold as to meddle with their coine. Thou proude, thou covetous, thou hautie citie of Hierusalem: *Argentum tuum versum est in scoriam*, thy silver is turned into what? into testions? *scoriam*,—into drosse. Ah seditious wretch, what had he to do with the minte? why should not he have left that matter to some master of policie to reprove? Thy silver is drosse, it is not fine, it is counterfait; thy silver is turned, thou haddest good silver. What pertained that to Esay? Mary he espied a peice of divinite in that policie, he threateneth them God's vengeaunce for it. He went to the roote of the matter, which was covetousnes; he espyed two pointes in it, that either it came of covetousnes, which became hym to reprove: or els that it tended to the hurte of the poore people, for the noughtynes of the silver was the occasion of dearth of all thynges in the realme. He imputeth it to them as a great crime. He may be called a master of sedition indeede. Was not this a seditious varlet; to tell them this to their beardes? to their face?"

At St. Paul's, Jan. 17, 1548, Bp. Latimer, in a sermon, was extremely severe in his censure of the appointment of bishops, and other ecclesiastics to lay offices, and more especially he pointed at their occupying situations in the Mint. He supposed that some of the following reasons might be assigned to excuse them being unpreaching prelates.

"They [the Bishops] are otherwise occupyed; some in the kinge's matters, some are ambassadours, some of the privy councell, some to furnish the courte, some are lords of the Parliament, some are presidentes, and some comptrolle of myntes. Well, well! Is this their duety? Is this their office? Is this their calling? should we have ministers of the church to be comptrollers of the myntes! Is this a meet office for a priest that hath cure of soules? Is this his charge? I would here aske one question: I would fayne know who comptrolleth the devill at home at his parish, while he comptrolleth the mynt? If the Apostles might not leave the office of preaching to be deacons, shall one leave it for mynting? I cannot tell you; but the saying is, that since priestes have been mynters, money hath bene wurse than it was before. And they say that the evilness of money hath made all things dearer."

than one month from the date of it, the current value of a testoon was cried down from 9*d.* to 6*d.*, the groat from 3*d.* to 2*d.*, the twopenny piece to one penny, the penny to the halfpenny, and the halfpenny to a farthing; which proves that the people had some cause for alarm.[1]

By the proclamation, the loss was thrown on individuals who held the base coins at the time; whereas it should have been borne by the nation at large. An almost similar proclamation was issued in 1843, whereby the temporary holders of the gold coinage, which by wear and tear had become light, were made to bear the loss.

Under these circumstances, great efforts were made for restoring the standard of the coinage.[2] In 1552, it was ordered that a pound of gold of *old standard* should be coined into 24 sovereigns of 20*s.* each, making £36 by tale; or 72 angels at 10*s.* each, and half-angels in proportion. And a pound of *Crown gold* of 22 carats fine and 2 alloy, into 33 sovereigns at 20*s.* or 132 crowns, making by tale £33.

A pound of silver of 11 oz. 1 dwt. fine and 19 dwts. alloy, was ordered to be coined into 12 crowns, 24 half-crowns, 60 shillings, 120 sixpences, 240 threepences, 720 pence, 1440 halfpence, or 2880 farthings, at £3 by tale. By this indenture, standard gold was to standard silver as 12 to 1, and Crown gold to silver as 11 to 1 (Lowndes, p. 47). This was the first coinage of threepenny pieces.

On King Edward's silver coins he is styled EDWARDUS VI. DEI GRATIA ANGLIE ET FRANCIE ET HIBERNIE REX. And the reverse has the words TIMOR DOMINI FONS VITE, and INIMICOS EJUS INDUAM CONFUSIONE. His gold coins bear the same style, but on the obverse or reverse is inscribed, JESUS AUTEM TRANSIENS PER MEDIUM ILLORUM IBAT, or SCUTUM FIDEI PROTEGET EUM, or PER CRUCEM TUAM SALVA NOS CHRISTE REDEMPTOR, or LUCERNA PEDIBUS MEIS VERBUM TUAM.

The expectations raised by the measures adopted for the restoration of the purity and integrity of the coinage were soon disappointed, for the king died on 6 July, 1553.

Mary, 1553—1558. On her accession, Mary announced *her intention* to restore the coins to their original standard of fineness of 18 ounces 2 pennyweights pure silver and 18 pennyweights alloy; but instead of that, she made it less fine than she found it. The last silver coinage of Edward VI. was 11 ounces 1 pennyweight fine, and 19 pennyweights alloy, but Mary's was 11 ounces fine and 1 ounce alloy. The proclamation was contradicted by this provision of the indenture for the new coinage. Before her marriage with Philip of Spain, she coined groats, half-groats, and pennies, the weight was in proportion to 8 grains to the penny, but the base pieces weighed 12 grains. Upon some of the groats she is represented in profile, to the left crowned, and wearing a cross suspended to her necklace. She is styled, MARIA D. G. ANG. FRANC. ET HIB. REGI,, and a pomegranite after her name serves as the mint mark. On the reverse is a shield and cross fleure, with the words VERITAS TEMPORIS FILIA, a pomegranite being placed after the first word. This motto, with the device of Time drawing truth out of a pit, was adopted by the persuasion of her Popish Clergy, in allusion to her endeavours to restore the Papal superstitions (as if they constituted the truth), which had been in a great degree suppressed by her predecessor Edward VI. (Hawkins, p. 144). In the first year after her marriage, the new coins exhibited the busts of Mary and Philip, face to face, with their titles.

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[1] Cowper has noted in his Chronicle the sufferings of the poor from these proceedings, as what they possessed lay chiefly in the current money; whereas he adds, "the richer sorte, partely by friendship, understanding the thing before hande, dyd put that kinde of money away; partely, knowyng the basenesse of the coyne, kept in store none but good golde and olde sylver that would not bryng anye losse."

[2] A letter of the Clerk of the Council, William Thomas, to King Edward VI., on the state of the coinage, contains the following passage:—

"If there be any quantity of gold remaining (as some men think small) it cannot come to light; because that like as the value of our silver money doth daily decay, so doth gold increase to such a value that, lying still, it amounteth above the revenues of any land. And he that shall live twelve months, shall see an old Angel (which by law was current for eight shillings), shall, in value and estimation, want little of twenty shillings of our current money, if provision for the redress of your Majesty's coin be not had." At length it became clear to the Counsellors of the King, that it was unnatural, and impossible to force permanently by law on a debased coinage a value above that which the precious metals, uncoined, bore in the ordinary course.

Elizabeth, 1558—1603, on her accession found the existing coinage in an unsatisfactory condition, and she had both the ability and the will to attempt to restore the silver money in relation to gold which it held in the time of her grandfather. On 27 September, 1560, a proclamation was issued, by which the base coins were ordered to be reduced to their true value as near as they might be. The shilling or testoon, which Edward VI. had reduced to sixpence, was now reduced to fourpence-halfpenny, excepting the very base testoons, which differed so much in value from the rest of the base testoons, that they should be taken only at their true value. But for the relief of such persons as held them, her Majesty was pleased to give pieces of good sterling money of fine silver, at the rate of 2¼*d.* for each testoon of twelve pence, and for every pound weight of them, three pence over of good money.

The coinage of the first three years[1] of her reign contained 11 oz. of pure silver and 1 oz. of alloy, but in her second year she restored the standard to its original fineness of 11 oz. 2 dwts. pure silver and 18 dwts. alloy, and in that state it has continued to the present time.

Queen Elizabeth ordained the pound weight of Crown gold to be coined into £33 by tale, and the pound of the old standard gold into £36 by tale, which is nearly of the same value as the gold coinage of 37 Henry VIII., when the pound weight of gold, 20 carats fine and 4 alloy, was coined into £30 by tale. It is clear, that here was no improvement in the value of the gold money; but the silver coinage was restored by coining the pound weight of standard silver into £3 sterling, which bears to £36 the value of one pound weight of standard gold, the same relative value as £1. 17*s.* 6*d.* bore to £22. 10*s.* in the reign of Henry VII.

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[1] Hitherto coins were struck with the hammer only, but in 1561, the third year of Elizabeth, a new process of coining was introduced by means of the mill and screw, but it did not come into use at the mint until 1662, when letters and grainings were first made on the edges of coins. The coins made by this process were called milled money. They are similar in type to the hammered money, but more neatly executed, rounder in form, and have their edges grained. It is a curious fact, that the hammered money should also have been coined at the same time, for a period of about fifteen years. It is well known that machinery, which is worked by a constant force, can always produce a greater mechanical effect in a given time than the arm of a man, which, from several causes, is subject to variation, whereas the former, besides a greater mechanical effect, always works with more accuracy. Not only in the act of coinage, but in all the useful arts, the extensive employment of machinery has been one of the chief causes of the superiority of the people of the British Isles. By means of their machinery, effects scarcely credible have been, and still are, constantly produced in every department of the arts. These effects have been mainly dependent on the abundant and constant supply of cheap coal, cheap iron, and cheap labour. If any one of these essential elements should fail, the natural effect will be the decline of England's superiority, and the mechanical arts will migrate to other countries where the necessary conditions for their success are available.

The opportunity for restoring the silver coinage of England was becoming more favourable, in consequence of silver in Europe becoming of less value than gold in commerce, from the increased supplies which were derived from the new world. The plan for restoring the standard of the coinage, begun by Edward VI., was carried out by Elizabeth, but not so completely as her professions and proclamations indicated. She permitted the Master of the Mint, by several commissions, to vary from the terms of his indentures, for the express purpose of coining money of less weight and fineness.

The title upon her coins was the same as that of Mary before her marriage, ELIZABETH DEI GRATIA ANGLIÆ FRANCIÆ ET HIBERNIÆ REGINA. The motto upon her silver coins is POSUI DEUM ADJUTOREM MEUM: upon her gold coins are found JESUS AUTEM TRANSIENS PER MEDIUM ILLORUM IBAT: or A DOMINO FACTUM EST ISTUD ET EST MIRABILE IN OCULIS NOSTRIS; or SCUTUM FIDEI PROTEGET EAM.

There was a large amount of the current money either clipped or counterfeit, and this caused both the relative value of gold and silver, and the rates of exchange between England and foreign countries, to be deranged. For the shilling of silver and the pound of gold, though somewhat less than their true weight, could be current in England at their legal value. But their relative value, and their value in foreign countries, varied less or more according to the diminution of their legal weight, and merchants could buy gold money of full weight with the current silver money of diminished weight, and the silver money of full weight with gold money of diminished weight. And during her reign, the money coined for Ireland amounted to 112,649 pounds weight, and its total value in English money was £94,577. 19*s.* 6*d.*, which at the rate of 16 pence Irish for every English shilling, comes to £118,222. 9*s.* 6½*d.* So that out of 112,649 pounds, only 2,977 pounds were sterling, and the remainder, being more than 40 parts to one of the whole amount, base alloy.

On a charter of incorporation being granted by Queen Elizabeth to the East India Company, the merchants were informed by the Privy Council, that her Majesty would not permit them to send the coins of the King of Spain or of any other foreign power to India, and that no coins should be imported but such as should bear her effigy on one side and the portcullis on the other. By virtue of a commission in 1600, a coinage was struck for the use of the Company in India. The weight of the pieces was to be regulated according to the just weight and fineness of the Spanish piastre or piece of eight reas, and the half, the quarter, and half-quarter of the same unit.

James I., 1603—1625. In the second year of his reign, James assumed the title of "King of Great Britain, France, and Ireland," and an indenture was executed 11 Nov. 1604, for a new coinage.

The weight, fineness, and type were nearly the same as the old, except on the smaller money the legends on the reverses were altered.

The crowns were similar in type to the former coinage. The indenture to Sir R. Martin ordered that one pound weight of gold, 22 carats fine and 2 carats alloy, was to be coined into £37. 4*s.* by tale, and the pound weight of silver into 62 shillings, or a proportional number of other coins.

A proclamation was issued on 16 Nov., in which it was stated, that great inconvenience had arisen from the Scottish coins of gold having been declared equal to the gold coins of England. It was not said that the Scottish coins were not worth so much of the English silver money; but because English gold coins in regard to silver coins, were not of the true proportion between gold and silver in other nations. This error had been the great cause of the transportation of gold out of the realm into other countries, because the gold coins of England were of more value in those parts than they were allowed to be current within the realm.

To remedy these and other inconveniences, new coins were struck both of gold and silver to be made of several stamps, but of one uniform standard and alloy, to be current in the kingdom of Great Britain; namely, a gold coin weighing 6 dwts. 10¾ grains, to be called the unit of 20*s.* value, and other gold coins, of the values of the half, the quarter, the fifth, and the eighth of the unit. The silver coins ordered were pieces of the value of five shillings, two shillings and sixpence, one shilling, sixpence, twopence, and one penny. All these were proclaimed and authorised to be the current coins of Great Britain.

In this year a very singular proclamation was issued dated 14 May, which exhibits strange proofs of ignorance of the principles of commerce, and equal boldness of assertion respecting the nature of the prices of the precious metals at that time; and a remedy was proposed by fixing the rates at which foreign gold and silver coins should be authorized to pass current. By another indenture of 1612, the pound weight of gold, old standard, was to be coined into £44 by tale, and a pound of gold, 22 carats fine, into £40. 18*s.* 4*d.* by tale.

In the year 1613, farthing tokens of copper were first coined, and allowed to pass in currency in his Majesty's dominions. Several coinages of gold and silver were ordered during his reign, and proclamations issued against the exportation of the precious metals, which do not appear to have been successful, as the merchants and goldsmiths gave higher prices for gold and silver than were given by his Majesty's mint. On the coins struck after his accession to the English throne, his style was Jacobus Dei Gratia Angliæ, Scotiæ, Franciæ et Hiberniæ Rex. In the second year, Angliæ Scotiæ were changed into Magnæ Britanniæ. His earliest English coins had on the reverse Exurgat Deus Dissipentur inimici. But after his second year, they were Quæ Deus conjunxit nemo separet; or Tueatur unita Deus; or Faciam eos in gentem unam, or Henricus rosas, regna Jacobus, in allusion to the two roses by Henry VII., and his own desired union of the two countries.

Charles I., 1625—1649, inherited the kingdom with his father's debts, and being engaged in war, he found his want of money exceeded all his means of supply. His first expedient was to enhance the values both of the gold and silver coins. The natural consequence of such a scheme would have been, that all the old money both of gold and silver would cease to circulate, and would become bullion, and if silver were enhanced more than gold, the gold would be bought up with silver. This scheme was quickly abandoned.

In 1626 he coined Crown gold of 22 carats fine at £44 by tale, and angel gold at 23 carats 3½ grains fine. The pound of silver was directed to be coined into £3. 10*s*. 6*d*. by tale; but it is doubtful if this indenture took effect. But there was another indenture with Sir. R. Harley by which a pound of old standard gold was to be coined into £44. 10*s*. by tale, and a pound of Crown gold of 22 carats fine into £41 by tale; one pound of silver of the old standard was to be coined into 62*s*. by tale.

The year 1628 was memorable for the great improvement which took place in the workmanship of the coins. Nicholas Briot, a native of Lorraine, offered his services to Charles I. They were accepted and he was made a denizen, and authorized to frame and engrave the first designs and effigies of the king's image.

During this reign it was the practice to pick out the heavy coins and melt them into bullion, and thereby were made considerable profits. In 1637 a complaint was exhibited in the Star Chamber against certain persons, for transporting money out of the kingdom, and giving prices for gold and silver above the prices of his majesty's mint and for melting down the coin into bullion. In the course of the trial it appeared that from 1628 to 1631 about £500,000 a year had been examined and £7000 or £8000 of heavy money was partly melted and partly sold unmelted. Heavy fines and imprisonment were inflicted on the offenders.

In 1640, the king's necessities became so urgent that in July he gave orders to seize upon the bullion to the amount of £40,000 which had been brought from foreign parts to be coined at the mint. He, however, was induced to release it on the condition of one-third of the sum being lent to him. On 24 April following, a petition signed by 20,000 citizens of London, remonstrated in these words, "that the stopping of the money in the mint, which, till then, was accounted the safest place and surest staple in these parts of the world, did still hinder the importation of bullion." About the same time the king bought up a large quantity of copper on credit, and sold it again immediately much under its value.

In a remonstrance which the Parliament presented to the king at Hampton Court on 1 Dec. 1641, they upbraided him with his violation of the public faith, as well as of private interest, in seizing the money and the bullion in the mint; and complained that the whole kingdom was like to be robbed at once in that abominable project of brass money.

These ways of getting money were only make-shifts.[1] It was proposed to debase the coin, and Sir Thomas Roe was called in, who being permitted to speak his opinion, declared that such a measure would intrench very far both into the honour and justice, and also into the profit of the king.

[1] In January, 1642, the king's household was reduced to so great want, that the Queen was obliged to coin or sell her plate for the supply of necessaries, as there was no money in the exchequer, nor in the power of the ministers of the revenue; as the issue of money from the customs, out of which the allowance for the weekly support of their majesty's household was supplied, had been forbidden by the Commons. About the middle of the year, all hopes of an accommodation between the king and his parliament seem to have been abandoned.

It is a fact, that during all the troubles of the king and his difficulties, no debasement of the standard took place, however rude his coins may have been in form and workmanship, after he had taken up arms against the forces of the Parliament.

The Commonwealth, 1649—1653. In 1642, the Commonwealth seized the Tower and the Mint, and proceeded to coin and issue money with the king's types and titles.

In 1649 it issued a silver coinage of various pieces of the same fineness as the 43rd Elizabeth. 11 oz. 2 dwts. pure silver, and 18 dwts. alloy; the weight being in proportion to $7\frac{23}{31}$ grains to the penny. On the obverse was a shield bearing the cross of St. George within branches of laurel and palm, with the legend on the four larger denominations, THE COMMONWEALTH OF ENGLAND. On the reverse were two shields, one bearing the cross of St. George, the other the harp of Ireland, with the legend GOD WITH US, and the date.

On 25 April the same year, the Committee reported, and the House resolved, that the inscriptions on the coins should be in the English language, that on one side should stand alone the English arms with the inscription THE COMMONWEALTH OF ENGLAND; and on the other side, the arms of England and Ireland, with the words GOD WITH US. These coins became a subject of standing jokes. In Hudibras will be found some wit on the subject. The Cavaliers took occasion from the legend on the coins to observe, that GOD and THE COMMONWEALTH were on different sides.

The Parliament also ordered a jury of goldsmiths to make two standard trial pieces, one of gold, 22 carats fine and 2 alloy, and one of silver, 11 ounces 2 dwts. fine and 18 pennyweights alloy; and by a resolution they were approved as standards for the new coinage.

It is remarkable that the pieces of date 1651, have the image and superscription of Cromwell, as Protector of England, Scotland, and Ireland, though he was not publicly invested with that title until 16 Dec. 1653. Coins were struck by the authority of Parliament, and they occur of all the intermediate dates from 1649 to 1660 inclusive.

A copper coinage was projected for the use of the poor. Pattern pieces exist of the dates of 1649 and 1651, but it is not known whether the design was ever carried into effect.

Cromwell understood the power of money. On learning the arrival of three ships from Hamburgh, having £300000 in silver on board, he ordered a detachment of soldiers to seize it. Having this money at command, he was emboldened to dissolve the Parliament.[1] The secret of his policy, like that of all other despots, appears to have been, the control of the army by money, the control of the people by the army, and thus governing the people by mere force of military power, he had at command the means of getting money, and the material of money wherever it was known to exist. Such methods of supplies might serve an emergency, but could not last long, as the sequel proved, by a restoration of the king and the laws for the protection of property.

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[1] "The Members of the Long Parliament, when it assumed the government, voted to each other, for his own private use, at first four pounds a-week, and afterwards, it was reported, distributed among themselves, out of the public treasury, about £300000 a-year. And, under the pretence of rewarding the godly for their services in the good cause, unbounded largesses were bestowed. Lenthal, the speaker, received £6000 at once, besides offices to the amount of £7730 a-year. Bradshaw, president of the high court of Justice, by whom the king was condemned, had the present of an estate worth £1000 a-year, and the king's house at Eltham, for the active part he took in that memorable transaction; and in free gifts to the saints, the sum of £679800 was publicly expended. The Parliament is also accused of suffering the most enormous frauds to be perpetrated with impunity. Instead of the public accounts being examined at the Exchequer, where peculation could with difficulty escape detection, every branch of the revenue, and every article of expense, was entrusted to committees of the

Oliver Cromwell, Protector, 1653—1658. It is more than probable that when the army ceased to be the servants, they became the masters of the Parliament, being entirely under the influence of Cromwell. And after he was proclaimed Lord Protector of the Commonwealth, he only exercised that authority which had before been entirely in his hands. The first act of his government declared in reference to the coinage what offences should be adjudged high treason.

There can be no doubt of Cromwell's intention to issue his coins for general circulation, otherwise it is difficult to account for his taking the opinion of his council on the form of them and the inscription. It is certain the coins of Cromwell were never the current money of the nation, because they are not named in the proclamation of Charles II. of 7 Sept. 1661, which forbade the currency of the coins of the Commonwealth, but made no mention of the coins of Cromwell as Protector, which, if they had been in circulation, could not have escaped notice. It is probable, that if he coined money as supreme ruler, he was prevented from uttering it, by the same considerations which deterred him from assuming the title of king.

Richard Cromwell was elected Protector within an hour after the death of his father, on 3 Sept. 1658. In answer to a petition in 1659, a warrant was issued for the coinage of copper farthings, and the powers of it were to last for 31 years. In the following month of May, Richard resigned the Protectorship, which prevented the carrying of this project into execution. It is a fact, that coins of the Commonwealth were struck bearing the date of 1659.

Charles II., 1660—1685. This monarch coined money on the same principles as his father, regardless of the improvements in the machinery, and in the manufacture of the coins which had been exhibited in the striking of pattern pieces during the Commonwealth and the Protectorate. He issued three distinct coinages of hammered money of the same fineness as that of the 43 Elizabeth.

In April 1662, a French artist was employed in the Mint to furnish the machinery, mills, &c , for coining, and to discover his secret unto

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house, who appropriated whatever sum they thought proper to their own private use. By these frauds, the Parliament was disabled from paying the army regularly. Its arrears amounted to £331000, and that mutiny, which proved the principal source of Cromwell's exaltation, was owing to the indignation with which the troops saw the members of the House of Commons rioting in wealth, procured by public plunder, whilst they who had fought their battles could hardly provide themselves with subsistence. They loudly complained, 'that Parliament bestowed upon its own members £1000 a-week out of the public treasury, whilst the soldiers' wants were great, and the people in the utmost necessity.' It is said, that the Parliament left about £500000 in the treasury, and stores to the value of £700000, when its authority was abolished by Cromwell; yet such was the expense of his administration, that he died indebted to the amount of £2474290, which, however, consisted chiefly in arrears due to the army and navy."—*Sinclair on the Public Revenue*, Vol. I., pp. 286—288, 3rd Ed. 1803.

his Majesty, and unto the Warden, Master, and Workers and Comptroller of the mint.

After the restoration of Charles II. in the year 1663, gold pieces of twenty shillings value were first coined, instead of the hammered broad pieces of gold which had been before in use of the same value. The guineas took their name from the gold brought from Guinea on the Western Coast of Africa by the African Company. As an encouragement to import gold for coinage, they were permitted to have the stamp of an elephant upon the coins made of the African gold. They were afterwards, by proclamation, made current as 21 shilling pieces, and ever so continued. A pound Troy of gold was coined into 44 guineas, each to pass for twenty shillings. Some of these guineas have under the King's head the elephant with a castle upon his back, others the elephant without the castle. There were likewise coined by the same company, gold pieces of forty shillings value, half-guineas, and pieces of five pounds, during the same reign.

In 1663, it was found expedient to modify the statutes which prohibited the exportation of bullion. Among the reasons assigned for this measure, there is one which observes, "It is found by experience, that money or bullion is carried in the greatest abundance as to a common market, to such places as give free liberty for exporting the same, and all of them are such as might have taught the legislature to see the absurdity of attempting to confine any kind of commodity within the kingdom by pains and penalties (as had been done hitherto); but it should seem that the last thing which all statesmen are willing to resign, is their weak and frequently pernicious interference with commerce, and other subjects beyond the province of parliamentary legislation."

In 1665, a coinage of copper farthings and halfpence was projected. They had on the obverse the king's bust laureat, with Carolus a Carolo; and on the reverse Britannia, with Quatuor maria vindico. This was the first coinage of real copper money; the regular coinage of copper for current coin of the realm takes its date from 1672.[1]

New patterns for the coinage were adopted, and the coins were struck by the improved process by machinery, and not by the hammer. On the obverse is the king's bust to the right, the head laureat with long hair, and his shoulders covered with a mantle of the antique style, and the words Carolus II. Dei Gra. or Dei Gratia. On the reverse, are the Royal arms upon four separate shields, crowned and arranged in the form of a cross, with the star of the order of the Garter in the centre. The inscription is composed of the date and the King's titles.

The crowns and half-crowns have the edges inscribed with Decus

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[1] The scarcity of money at this time formed one of the topics of Lord Lucas' severe speech against the government on 22 Feb. 1670, on the second reading of the Subsidy Bill, in the presence of his Majesty. "And it is evident," he said, "there is a scarcity of money; for all the Parliament money, called breeches money has wholly vanished; the king's proclamation and the Dutch have swept it all away, and of his now majesty's coin, there appears but very little; so that, in effect, we have none left for common use, but a little *lean-coined money* of the late three former princes; and what supply is preparing for it, my Lord? I hear of none, unless it be of copper farthings, and this is the metal that is *to vindicate*, according to the inscription on it, *the dominion of the four seas.*"

ET TUTAMEN, and the year of his reign with one or two exceptions. This inscription was placed round the edge to prevent clipping. Evelyn states that it was suggested by himself to the Master of the Mint, having observed it in a vignette in the Cardinal de Richelieu's Greek Testament. The inscription round the edge began to be dated with the year of his reign, but different years of the reign sometimes appear on coins of the same date. The crown of 1663 is inscribed DECUS ET TUTAMEN. ANNO REGNI XV. On that of 1667 is found DECIMO NONO, and subsequently the Roman numerals were changed to the corresponding words.

James II., 1685—1688. The short and unhappy reign of King James II. was in almost every respect eminently disgraceful, and in no single instance more so than in the state to which he at length reduced the coinage in his kingdom of Ireland. How different was the conduct of his father Charles I., who in his extreme distress never debased the standard of the coinage of the realm. The money of England escaped violation, for James was forced to abandon that kingdom before his necessities became very urgent, and only the coinage of his first year was of the same standard as his brother's. On 23 December, 1688, he was compelled to abdicate the throne, on account of his unscrupulous attempt to force the Roman Catholic religion upon the nation in violation of his Coronation oath, and he retired into France. In the following March he made an attempt to recover the Crown, and on the 12th, with about 5000 French troops, he landed at Kinsale. On the 24th he entered Dublin in a triumphant manner, and issued a proclamation, raising the value of the current coins, the guinea in gold to 24 shillings, and the crown in silver to *5s. 5d.*

But all his expedients failed to procure a sufficient supply of money, he coined brass and copper sixpenny pieces to meet his immediate necessities, and shortly after, copper shillings and half-crown pieces. Some of them are known for every month from June, 1689, to April, 1690, inclusive. To supply the mint with metal for this degraded coinage, two brass cannons from the Court of Dublin Castle were delivered to the Commissioners of the mint. And it appears that they not only bought old brass, old broken bells, and pewter and copper, but actually pillaged the metal utensils from the kitchens of the citizens of Dublin, because they found it difficult to get a sufficient supply of copper or brass for the mint.

These brass and copper monies were found insufficient for the expenses of his army. On 1 March money of white mixed metal was coined into pieces about the size of a shilling and a sixpence, to be current for a penny and a halfpenny respectively. These coins had on one side the head of the king with JACOBUS II DUS. DEI GRATIA; and on the other side a piece of prince's metal fixed in the middle with the impression of the harp and crown, and the inscription MAG: BRI: FRA: ET: HIB: REX, with the date. A further coinage of white metal for crown pieces was ordered to be current on the 21 April. On 15 June the brass and copper half-crowns were called in, and restamped with the die which was used for the white metal crown pieces, and then re-issued at the value of *5s.*

There is an account of this coinage preserved, from which it appears that the weight of metal used was 389724 lbs. 2¼ ozs. Avoirdupois, which at *4d.* a pound (valued by the workmen at the mint) gives the real value at £6495. *8s. 4d.* This metal was coined and issued as current money; and the total value imposed upon the coins by arbitrary power was no less than £1596789. 0*s.* 6*d.* sterling.[1] If to this amount be added the increase by raising the half-crowns to the value of crowns, and the reduction in the weight of the large shillings and small half-crown pieces, these additions on a very moderate computation would raise the above amount to £2163239. 9*s.* as the produce of £6495, the real value of the metal. There might also be some further additions made to this large sum, if it were known exactly what proportions were coined into copper half-crowns and restamped into crowns, and also what sums were coined into white mixed metal crowns, and into pewter pennies and half-pennies. A prodigious sum of money was raised by this scheme in a short time, exceeding £180000 a month. Of all this money, when that unhappy prince fled from Ireland, there was little found left in the mint, not above £22489, of metal value £642, as appears from the accounts of Lord Coningsby, Vice-Treasurer of Ireland, and of which he rendered an account.[2]

In this wretched money the Popish soldiers were paid their subsistence, and the Protestant tradesmen and creditors were obliged to receive it for their goods and debts, and it was computed that they lost upwards of £60000 a month by this cruel stratagem. The Governor of Dublin, the Provost-Marshal and his deputies, threatened to hang up all who refused it.

The records of history in no former age of the civilized world exhibit a parallel to these proceedings of King James II. and his Roman Catholic advisers and adherents. By the force of his Royal prerogative, he invested base metal of the real value of four pence, with the current value of £10 sterling.

William III., 1689—1702. The abdication of James II. opened the way for King William and Queen Mary to the throne of England. In the first year of this reign a coinage was ordered to be continued of the same kind, weight and fineness, as had been ordered in the first year of King James II.

Crowns were issued with the busts of the King and Queen on the obverse towards the right, surrounded by the words GULIELMUS ET

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[1] The amount stands thus:

Weight of metal			Value imposed on coinage.		
Avoir. lb.	oz.		£.	*s.*	*d.*
62,422	2¼	coined into large shillings	245,879	17	0
110,308	15	,, half-crowns	443,498	10	0
172,731	1½	,, large shillings and half-crowns .	689,378	7	0
14,080	3	,, small sixpences	49,042	6	6
8,914	11¾	,, small shillings	41,800	0	0
21,267	0¾	,, small half-crowns..............	127,200	0	0
lbs. 389,724	2¼		£1,596,799	0	6

[2] Accounted for as thus stated (Simon, p. 62).

	Imposed value.				real value.		
	£.	*s.*	*d.*		£.	*s.*	*d.*
17292 copper crowns	4,323	0	0	valued at 1*d.* each	72	1	0
126503 large half-crowns....	15,812	17	6	,, 1*d.* each	527	1	11
2489 small crowns........	311	2	6	,, ¾ ,,	7	15	6
9043 large shillings	452	3	0	,, ½ ,,	18	16	9
4757 small shillings	237	17	0	,, ¼ ,,	4	19	1¼
6000 copper sixpences	150	0	0	,, ¼ ,,	6	5	0
4808 pewter crowns	1202	0	0	,, ¼ ,,	5	0	2
	£22489	0	0		£641	19	5¼

Maria Dei Gratia. On the reverse, the four shields are arranged crosswise as before, but having the shield of Nassau in the center and round it the numerals 1691, and in the angles W and M interlinked, and the continuation of the inscription, Mag. Br. Fr. et Hi. Rex et Regina.

On the obverse of the half-crowns was the same as on the crowns. On the reverse, a square shield, crowned, bearing in the four quarters the arms of England, Scotland, Ireland, France; Nassau on an escutcheon of pretence. This is probably the only instance of the arms of France being placed in the fourth quarter. The inscription is the same as on the crowns, 1689. The coinage also included shillings, sixpences, groats, threepenny pieces, half-groats and pennies.

On 14 June, 1690, William landed in Ireland, and having defeated the army of James, at the Boyne on 1 July, he encamped near Dublin, and issued a proclamation to stop at once the mischievous effects of the debased money. This was done by reducing the debased coins to the value of the metals of which they were composed. In the year 1694, a license was granted to coin 700 tons of copper into halfpence and farthings, at the rate of 22 pennies for every pound weight of copper.

On 28 December, 1694, Queen Mary died, and after this event the king's effigy and style only were placed on the coins. At this time, the trade of the nation suffered from the excessive debasement of the money, both by clipping and counterfeiting, and by the exportation of the good money.[1]

In the Session of Parliament, 1695, an Act was passed to suspend for a time the coinage of guineas, and the importation of gold coin was also prohibited. These measures, in conjunction with the Bank

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[1] In Evelyn's Diary, 1694—1696, are found the following passages .—"Many executed at London for clipping money, now done to that intolerable extent, that there was hardly any money that was worth above half the nominal value."

"Great confusion and distraction by reason of the clipped money."

"Want of current money to carry on the smallest concerns, even for daily provisions in the market."

"Add the fraud of the bankers and goldsmiths, who having gotten immense riches by extortion, keep up their treasure in expectation of enhancing the value. Duncomb, not long since a mean goldsmith, having made a purchase of the late Duke of Buckingham's estate at near £90,000, and reputed to have near as much in cash."

"So little money in the nation that exchequer tallies, of which I had for £2000 on the best fund in England, the Post office, nobody would take at 30 per cent. discount."

In the year 1696, merchants and others petitioned the House of Commons, and stated that by the artifice of brokers and others, guineas were advanced from 21 to 30 shillings, and that they could get no money for their goods unless they received the coins at that rate; and that they could not pay them away again at the Custom House or on foreign bills without loss. On the other hand, several graziers and others petitioned against lowering the price of guineas, alleging they had for about twelve months received them at 30 shillings. A third petition from several merchants stated that the use of guineas, on account of the badness of the silver coins, had raised gold about 40 per cent. in value above the proportion of gold to silver in any other part of Europe. A fourth petition represented that they had of late been imposed upon by bankers and goldsmiths, to whom they had been compelled to pay the guineas under 29 shillings, although they had received them at 30 shillings a piece, and prayed the price of guineas might fall gradually. The House resolved upon a reduction of the value of guineas, and by a clause in an Act, passed in the same session, the current value was reduced to 22 shillings, on account of the prejudice which trade had sustained from the of England, then recently established,[1] struck at the root of the monopoly of the precious metals. For when men were deprived by law from the possibility of obtaining gold, no man could be compelled by law to make payments in gold; and bank-notes, bills of exchange, and silver would constitute the only existing currency. This suspension of payments in gold produced the effect, that the value of gold declined rapidly.

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uncertain value of coined gold, which had encouraged certain evil-disposed persons to raise and lower the same, to the great prejudice of the landed men of the kingdom.

Sir Dudley North, in the time of William III., thus declared his opinion against the renewal of the Free Coinage Act:—

"I call to witness the vast sums that have been coined in England, since the free coinage was set up. What is become of it all? Nobody believes it to be in the nation; and it cannot well be all transported, the penalties for doing so being so great. The case is plain, the melting pot devours it all; the latter, because that practice is so easy, profitable, and safe from all possibility of detection, as every one knows it is; and I know no intelligent man who doubts but the new money goes this way. Silver and gold, like all other commodities, have their ebbings and flowings. Upon the arrival of quantities from Spain, the mint commonly gives the best price for it: that is, coined silver for uncoined silver, weight for weight, wherefore it is carried into the tower and coined. Not long after there will come a demand for bullion to be exported again. If there is none, but all be in coin, what then? melt it down again; there's no loss in it, for the coinage cost the owners nothing. Thus, the nation has been abused, and made to pay for the twisting of straw for asses to eat."

[1] The Bank of England received its charter of incorporation in 1694, when the whole of its original capital was lent to the Government. In 1750, it began to issue notes for £10, its previous issues up to this date having been for notes of £20 and upwards. In 1793, notes for £5 were first issued, and in 1797, notes for £1 and £2 were put into circulation as a relief under the Bank Restriction Act, which restrained the Bank from making payments of its notes in gold and silver. At the end of the year 1813, the amount of notes in circulation was nearly 24 millions, the price of gold was £5. 10*s*. an ounce, and the depreciation of the Bank notes in consequence was between 29 and 30 per cent. At the end of the next year the circulation of notes had increased to about 28½ millions, but gold had fallen to £4. 6*s*. 6*d*. an ounce, so that the notes were only depreciated to about 10 per cent. The rise in the value of notes alone, 19 per cent., was chiefly caused by the large quantity of gold brought into the country on the reopening of commerce after the peace of 1814. In 1817, the Bank gave notice, that all notes of £1 and £2 of dates prior to 1816 might be received in gold, and in 1819 an Act was passed for the gradual resumption of cash payments, but provided that the Bank Restriction Act should be continued in force until 1 February, 1820. From that date till the following 1 October, the Bank was required to pay its notes in standard bullion, at the rate of £4. 1*s*. an ounce. From 1 October, 1820, to 1 May, 1821, at the rate of £3. 19*s*. 6*d*. an ounce, and after that date at the rate of £3. 17*s*. 10½*d*. an ounce; and on 1 May, 1823, gold coins of the realm might be demanded. The Bank, however, anticipated these provisions, and began on 1 May, 1821, the payment of their notes in specie. On the renewal of the Bank Charter in 1833, a provision was made for any other Bank beyond 65 miles from London to issue notes payable on demand. A clause in the Act also provides, that notes of the Bank of England shall be a legal tender in any part of England. This relieves the Bank of England from providing bullion to meet any run upon country banks which can pay the demands on them by Bank of England notes instead of specie.

The establishment of the Bank of England effected an important change in the monetary system of the country. As bank notes could be paid for taxes, customs, and in all other commercial payments as gold and silver, they were soon found to be as serviceable, and to diminish the necessity for gold, and consequently to diminish its value. It also altered the relative value which had previously subsisted between gold and silver. The Report of the Committee of Secrecy, made in August, 1832, contains the evidence and experience of practical men on the subject and the principles of banking.

An Act was passed to commence on 25 March, 1696, to continue seven years, to raise a tax of £1200000 by a duty on dwelling-houses, for supplying the deficiencies of the clipped money then in circulation by a new re-coinage. The coins were found to be so diminished as to weigh less than half the legal weight. The coinage of the provincial mints bears the dates of 1696 and 1697, from which it may be inferred, that the new coinage was completed in two years. The mint at the Tower continued the coining of large sums for about two years longer.

It appears that in making up the accounts in 1699, the actual quantity of silver money coined at the Tower mint was £5091121. 7*s.* 7*d.* and in the provincial mints £1791787. 12*s.*, making a total of £6882908. 19*s.* 7*d.* If we suppose only two-thirds of this sum were worn or clipped coins, which were diminished nearly one-half, and received at the exchequer in payment of the public revenues, and at their nominal value, the loss to the public, on this account, would amount to £2294302. 19*s.* 10*d.* If we suppose the remaining third cost the public no more than sixpence an ounce, or about 10 per cent., this gives a further loss of £229430. And adding £179431. 6*s.* the mint charges, the three sums amount to £2703164. 5*s.* 10*d.*, the total loss to the public.

Anne, 1702—1714. On the death of William III., Anne, the second daughter of James II., succeeded to the Crown. The beauty of the coinage of this reign was very greatly improved. By the articles of union of the two kingdoms of England and Scotland, which had been agreed to on 22 July, 1706, and which were to take effect on 1 May, 1707, it was provided that the coin should be of the same standard and value throughout the United Kingdom, as it was at that time in England, and that a mint should be continued in Scotland, under the same rules as the mint in England. An alteration was made in the Royal Arms on the coins; England and Scotland were then impaled on the first and third quarterings, France placed in the second, and Ireland in the fourth. The Queen's bust is the same, both upon the gold and silver coins; her style was the same as in the last reign.

George I., 1714—1727. The coinage of George I. was of the same species and value as that of Queen Anne, but to his style on the reverse were added his German titles with FIDEI DEFENSOR,[1] which then for the first time appeared on the coins, although it had been constantly used in the style of our monarchs from the time of Henry VIII. The adoption of the words FIDEI DEFENSOR on the coinage of George I. cannot be understood to mean that the King of Great Britain is a defender of faith in the dogmas of the Romish Church, as was Henry VIII.[1] These words must be taken in reference to the reformed faith, and not to faith in the Romish dogmas, which are described in the Articles of the Reformed Church of England as "blasphemous fables and dangerous deceits" and as "repugnant to the Word of God."

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[1] This title of "Fidei Defensor" was conferred in 1521 by Pope Leo X. on King Henry VIII., on account of his work in Defence of the Seven Sacraments of the Romish Church against Martin Luther, the German Reformer. The identical copy of this work on the Seven Sacraments, presented to Leo X., may be seen in the Library of the Fitzwilliam Museum, at Cambridge. On its binding are impressed the Tudor Arms, and the undoubted sign manual of Henry VIII. may be read inscribed both at the beginning and end of the work. About two years after Henry VIII. had received this distinguished title, he ordered the learned men of his time to search the Records of the Realm, and to ascertain on what grounds the Bishop of Rome assumed and exercised authority in the Realm of England. After a long and accurate search in ancient records, they reported that "The

In 1717, there was a coinage of halfpence and farthings issued from the Tower. The pound of copper Avoirdupois was coined into 28 pence instead of 21 as in the time of King William, and about 213½ tons were coined, giving the amount of £46,000 sterling.

In the same year after an address of the Commons, a Royal Proclamation was issued, which declared that the relative value of gold and silver in the current coins, was in England greater in proportion than in other nations of Europe; and this had been the chief cause of the export and lowering the species of the silver coins. It was in consequence ordered that no person should either give or receive any of the gold coins of England called guineas, at a higher rate than twenty-one shillings, and in like proportion for the half-guinea and smaller pieces. The guinea itself was originally coined of the value of twenty shillings, but had been current at twenty-one shillings and sixpence. This order was intended to bring the guinea nearer to its value in silver bullion, which Sir Isaac Newton had stated to be twenty shillings and eightpence.

The House of Lords also in Committee took into consideration the matter of gold and silver coins.[1] Lord Stanhope imputed the scarcity

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Pope had no authority at all in England, either by the Laws of God, or by the Laws and practice of the Primitive Church, or by the Laws of the Land." On this Report was founded the Act 24 Henry VIII., c. 12, which declared the supremacy of the Crown, and the independence of the Realm of any foreign prince, potentate or prelate. Next the Act 28 Henry VIII., c. 10, with some other Acts, abolished all usages in England which had been founded on no other authority than the Papal decrees.

In "The Cambridge Documents," published by Dr. Lamb, late Master of Corpus Christi College, will be found an account of a public disputation held at Cambridge on the question—"Whether the Pope had granted him by God in the Scriptures any greater authority or power in this kingdom than any other foreign bishop?" The question was determined in the negative, and this decision was confirmed by the votes of the Senate on 2 May, 1534.

The king, however, was so leavened with the old superstition, that he lived and died a devoted adherent to the dogmas of the Church of Rome, which he had defended in his work on the Seven Sacraments. It was the same Sovereign, in the 31st year of his reign, who signed the Act for the suppression of Monasteries, and the Act of the Six Articles, which latter imposed submission to the Romish dogmas under the pain of death by burning, and, as in the case of high treason, the forfeiture of lands and goods.

[1] In a Report of Sir I. Newton in 1717, then master of the mint, it appeared that in the last year of William III., the French Louis d'or was current for 17*s.* 6*d.* when its real value was only 17*s.* 0¾*d.* An order was issued that the Louis d'or should be current at 17*s.*, and therefore they were brought to the mint as bullion, and the coinage from the bullion produced £1,400,000. In this case the advantage of 5¼*d.* for each Louis d'or brought them very largely into the kingdom, and the loss of ¾*d.* on each, drove them out of circulation. On another occasion, Portuguese moidores passed for 28*s.*, and they abounded in the country. A Report was made that they were intrinsically worth 27*s.* 7*d.* each, and their currency was ordered at 27*s.* 6*d.* A complaint was raised and the moidores disappeared, so that a profit of 5*d.* on each brought them into the kingdom, and the loss of 1*d.* sent them out again. It may also be noted, before the issue of the new silver coinage in the time of George IV., an immense number of francs and half-francs were in circulation as shillings and sixpences. The intrinsic value of the franc was 10*d.*, and the gain of 2*d.* on every franc passed as a shilling, brought some millions into circulation in this country. On

of silver to the increasing luxury in relation to silver-plate, the vast exports of bullion to the East Indies, and the clandestine trade of exporting gold and silver to Holland and other parts.

In 1723, a patent was granted to W. Wood, Esq., to coin halfpence and farthings to supply the want of small money in Ireland, for the use of such persons as would, without compulsion, voluntarily receive it. The patent was for 14 years. The copper was limited to 360 tons; 100 tons to be issued in coins the first year, and 20 tons annually for the remaining 13 years, under a comptroller appointed by the Crown. A pound of copper was to be coined into two shillings and sixpence by tale. The sum of £800 a year was reserved for the king, and £200 for the comptroller, to be paid by the patentee.

This measure was extremely unpopular in Ireland, and the prejudices of the people were so strongly raised against it by Dean Swift and others, that the patentee was compelled to resign his patent.[1] On 18 August, the King in Council was pleased to direct that the coins already made to the value of about £17,000, and as much more as would make up the sum to £40,000, should be permitted to be current according to the terms of the patent. Mr. Wood was charged with fraud and deceit in executing the powers granted to him as patentee. A Committee of the Lords of the Privy Council was

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the issue of the new coinage all foreign coins were refused, and only old English coins were received in exchange for the new shillings and sixpences.

Mr. Fleetwood, in a sermon before the Lord Mayor, on Gen. xxiii. 16, pointed out the mischiefs arising from debasing the coins either in weight or fineness, and the wickedness of the practice, as being a fraud upon every person who received the coins so debased. He pointed out the calamities which would ensue, that a time would come when the money would be no longer current, but at its just weight and fineness. Then every family would be a loser, but the loss would fall most severely on the poor.

[1] By misrepresentations and the delusive force of a sermon of Dean Swift, and his Drapier's letters, he brought to the test of experiment, an impudent and unprincipled assertion of his own, that were he permitted to write whatever he pleased, he would engage to write down any government in a few months.

Swift attacked the report with sophistries and misstatements, which were well calculated to mislead minds already prejudiced against the measure; and when a proclamation was issued offering £300 reward for the discovery of the author of the Drapier's fourth letter, and a bill of indictment was preparing against the printer of it; Swift published "Seasonable advice to the grand jury," in which by similar modes of arguing he called upon them not to find the bill. A copy of the pamphlet was distributed, on the evening before the trial, to every person on the grand jury; and thus by the very means which Swift himself had so strongly reprobated when used by the Committee of the Privy Council, namely, by prejudging the case, he accomplished his purpose, and the bill was not found. From this time, Swift was considered the Saviour of Ireland, and Wood was ridiculed in ballads and executed in effigy, and at last obliged to resign his patent, and (as Mr. Leake expresses it) *for the satisfaction of the people of Ireland!*

Lord Chesterfield has truly remarked that "every numerous assembly is a mob, let the individuals who compose it be what they will. Mere reason and good sense are never to be talked to a mob; their passions, their sentiments, their senses, and their seeming interests are alone to be appealed to." On this principle Swift wrote, and his writings were, in the instance before us, eminently successful. But the triumph attending such success is shortlived, while the infamy of it is

Dean Swift appears to have forgotten, or wilfully ignored the fact, that it is the prerogative of the Sovereign of a State or the supreme power, and not that of the people, to coin money, and to give authority for making it current. And, further, to impose such a value on foreign coins, as he may appoint, to make them current in his kingdom. And it may be added, that this prerogative has ever been by English sovereigns guarded with extreme jealousy.

appointed to investigate the truth of these charges. From their report it appeared that no papers nor evidence from individuals, which might be necessary to support the charges and objections against the patentee, could be obtained from Ireland. The patent, nevertheless, was cancelled, and for the loss which Mr. Wood had sustained, an annual pension of £3,000 was awarded, to be continued for eight years.

A Commission was appointed, consisting of Sir Isaac Newton and others, to make trials and assays of Mr. Wood's copper coinage, and they reported that, though all the coins were not exactly sized, yet when taken together they exceeded the weight required by the patent. The copper was of the same goodness and value as that which was coined for England, and that Mr. Wood's coinage of halfpennies and farthings exceeded in weight and fineness those which had been coined for Ireland in the reigns of Charles II., James II., and William and Mary.

George II., 1727—1760. No alterations were made in the form or value of the coinages during this reign, nor in the style of the coins. In 1729, the first coinage was ordered to be made of 46 halfpence or 92 farthings out of one pound of copper Avoirdupois.

In 1732, the old hammered gold coins were called in, and the officers of the Mint were authorized for one year to receive them at the rate of 81*s.* for one ounce Troy, and to recoin them into other current money.

About 1736, great inconveniences were experienced in Ireland by the want of good copper money, and on the humble request of the Lords Justices and Council, an order for the coinage of halfpence and farthings was issued.[1] At the Mint in the Tower of London fifty tons of copper were coined, of which five-sixths were made into halfpence, and one-sixth into farthings. The pound Avoirdupois of copper was cut into 52 halfpence or 104 farthings. His Majesty's effigy was impressed on the obverse with GEORGIUS II. REX only, and on the reverse, the Irish harp crowned with the inscription HIBERNIA, with the date of the year. His Majesty was pleased to direct that the expenses of this coinage, and of the transmission of it to Ireland, should be paid by His Majesty's Vice-Treasurer, and any profits that might remain should go into the public revenue of Ireland.

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[1] The omission of Dei Gratia on these coins was noted in the *Gentleman's Magazine*, for June, 1737:—

No Christian kings that I can find,
  However match'd or odd,
Excepting ours, have ever coined
  Without the *Grace of God.*

By this acknowledgment they shew
  The mighty King of kings,
As Him from whom their riches flow,
  From whom their grandeur springs.

Come then, Urania, aid my pen,
  The latent cause assign,—
All other Kings are mortal men,
  But GEORGE, 'tis plain, 's divine.

In the number for the next month appeared the following reply:—"To the author of the epigram on the new Irish halfpence":

While you behold th' imperfect coin,
  Received without the *Grace of God*,
All honest men with you must join,
  And even Britons think it odd.

The Grace of God was well left out,
  And I applaud the politician;
For when an evil's done, no doubt,
  'Tis not by *God's grace*, but permission.

In the next year a Proclamation was issued, which declared at what rates the several pieces of gold in circulation should be current.

Besides the English gold coins, the guinea and the half-guinea, there was in circulation a large number of the gold coins of France, Spain, and Portugal. It was ordered in case any of the coins named in the proclamation were found deficient in weight, that twopence should be allowed for every grain wanting of the true weight in each coin, one penny for half a grain, and one halfpenny for a quarter of a grain, and with this allowance the coins were to be received as of full weight.

1742. About this time offences against the coinage laws had greatly increased, both by filing and sweating the gold coins, as well as by counterfeiting them in base metal and gilding them, so that a remedy became urgent. The £400 a year allowed by an Act of the ninth year of Queen Anne, for several years had proved insufficient for the expenses of prosecuting the offenders. An Act was passed (15 Geo. II. c. 28.) by which an additional sum was allowed for the expenses of such prosecutions, but the amount was not to exceed £600 in any one year.

George III., 1760—1820. On the accession of George III., the grandson of George II., the coinage was still found to be in a very imperfect state, and during his long reign, measures were taken from time to time to amend its defects, and to render it sufficient for the requirements of the nation.

In the year 1771, the gold coins were found to be in a very defective state; three-fourths of the silver coins were base, and the copper was as bad as the silver. The offences against the coinage had so increased, that £1136. 19*s*. 10*d*. was allowed in addition to the £600 allowed for the expenses attending such prosecutions in the year 1770.

In 1774 an Act was passed, that the deficient coins should be called in and recoined, and that £250,000 should be granted towards the expenses. And that in future, the currency of gold coins should be regulated by weight as well as by tale, as was conformable to the ancient laws of the kingdom. The regulation thus established of weighing the gold coins, has been the means of preserving them at nearly the same state of perfection to which they were then brought. Besides the sum of £250,000 granted in 1774, other sums were granted in the three following years amounting to £267,320. In the year 1792, silver coins and bullion largely disappeared by the policy of the French, who exchanged their assignats for all they could procure. So rapidly they acted in this business, that not less than 2909000 ounces of silver were purchased with the assignats, and sent into France. It had before been ascertained, that a recoinage of silver was necessary, and in 1797, the coinage transactions formed a strange anomaly in the history of the mint. The deficiency of silver coins was attempted to be supplied by the issue of Spanish dollars, countermarked on the neck of the bust with the mark of the king's head, used at Goldsmith's Hall for distinguishing the plate of this kingdom. This scheme was abandoned in less than seven months from the date of its adoption.

By statute in 1797, the 37 Geo. III. c. 45, 91, the Bank of England was restricted from making payments in cash. This measure was only a palliation, and not a cure of the evil which produced it; and experience has made it doubtful, whether a recoinage of the gold money of such a weight as might have rendered it unprofitable either to melt or to export it, would not have been more expedient.

In the same year a contract was entered into with Mr. Boulton of Soho, near Birmingham, for the coinage of 500 tons of copper moneys in penny and two penny pieces. The penny piece was to weigh one ounce Avoirdupois. Each piece was to have on one side the king's effigy, with his name or title, and on the reverse the figure of Britannia, sitting on a rock in the sea, holding a trident in her left hand, and a branch of olive in her right, and the date of the coinage.

Two years after this a proclamation was issued for the currency of Mr. Boulton's new coinage. Owing to an unexpected rise of copper, the Privy Council allowed 36 instead of 32 halfpenny pieces to be coined out of the pound of copper.

1801. Upon the union of Great Britain and Ireland, it was declared by proclamation on 1 January, 1801, that His Majesty's royal style and title should be Georgius Tertius, Dei Gratia Britanniarum Rex. Fidei Defensor; and that the arms of the United Kingdom should be, quarterly, first and fourth, England; second, Scotland; third, Ireland; and on an escutcheon of pretence the arms of His Majesty's dominions in Germany.

In the year 1804 the Bank of England was authorized to issue silver tokens of five shillings value, and 1211484 were issued during the year. These coins had on the obverse the king's bust laureat, with the circumscription GEORGIUS III. DEI GRATIA REX. On the reverse was Britannia seated under a turreted crown, holding an olive branch in her right hand and resting the left on a shield and spear. On her left side was a cornucopia, and a bee-hive on her right, with the inscription BANK OF ENGLAND, 1804. FIVE SHILLINGS DOLLAR.

From a sudden rise in the price of copper, the greater part of the penny and twopenny pieces disappeared, because they were worth when melted down nearly one-third more than their value in coins.

On 7 May, 1806, a new coinage of penny, halfpenny, and farthing pieces was made current on the same terms as those issued in 1799. There were 150 tons of copper coined into penny pieces, 24 to the pound Avoirdupois; 427½ tons into halfpenny pieces, 48 to the pound; and 22½ tons into farthings, 96 to the pound. All these were of the same type and form as those of 1797, but of less weight.

In 1810 a Committee of the House of Commons was appointed to inquire into the causes of the high price of bullion and its effect on the credit paper currency. This report declared that the evil arose from the excessive issue of Bank of England notes since 1797; that a rise in the price of gold, and a fall in foreign exchanges, and a general rise in prices of all commodities, would always be the effect of the undue quantity of the currency of a country, which is not exportable to other countries; and that no sufficient remedy for the present evil, or security against its recurrence existed, except in the repeal of the law which suspended cash payments. Other causes might be stated, one of which is the law which fixes bullion when coined to a certain value, but which has no power over uncoined bullion, and, therefore, leaves it, like other commodities, to find its price according to the supply and demand.

As gold coins are fixed at £3. 17*s.* 10½*d.* the ounce, it is singular that the Committee should be surprised that the ounce in coin was not equal in value to the ounce in bullion, which latter was worth £4 and upwards; and that they should conceive such inequality in value to have been occasioned by a superabundance of paper money, when they might have seen that, if the coins were free from restriction, they would become of the same value as standard bullion.

The Committee assumed that the gold coin is *the measure of value*, and on this assumption founded the most essential points of their Report. But a *measure* implies something fixed and unchangeable, which the material of coins cannot be, so long as it is a subject of traffic. The truth is, the pound sterling is our actual measure in this kingdom, and the coin is only the instrument by which that measure is applied. So long as it remains, or is supposed to remain, precisely equal to its prototype, so long only is it an accurate substitute for it. Whenever it exceeds, or falls below the value of the pound sterling, it equally becomes an incorrect resemblance of it.[1]

The debate in the House turned on the question, whether the bank notes were depreciated or the guineas enhanced in value. If the dividends payable for the interest of the National Debt were made the test, the bank note was not depreciated, but the guinea was raised. If coin or any other mercantile commodity be the test, it will appear that the guinea was not enhanced, but that the bank note was depreciated. This dilemma was not considered, but after a debate which lasted seven nights, the majority of the House voted that the bank note was not depreciated,[2] but that it was highly important that the restrictions on cash payments at the Bank of England should be removed whenever it was compatible with the public interest.

It is curious to remark, that the Bank of England issued a notice on 18 March, 1811, that the price of silver had risen so much since the issue of bank dollar tokens at 5*s.* each, as to make them worth more as bullion than as coins. It was deemed expedient, in order to prevent them from being withdrawn from circulation, that they should in future be current at the increased value of 5*s.* 6*d.* each. In the year 1812, an Act was passed to amend that of 51 Geo. III., c. 127, respecting the gold coins and the notes of the Bank of England.[1]

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[1] From 1797, the circulating medium has been chiefly carried on by a credit currency of exchequer bills, bank notes, and promissory notes. The manufacturer having employed his capital in the production of goods, sold them to the merchant, and received a promissory note payable at a fixed date. The country banker discounted this note with his own local notes payable on demand. With these notes the manufacturer could pay his workmen and produce other goods, and his workmen could with them procure necessaries. When the promissory note became due, it was discharged by the produce of the sale of the goods. This method of local currency dispensing with the precious metals, and depending on credit, became the means of very largely increasing the industry of the people. The system, however, is subject to serious drawbacks and dangers, which have not failed to cause both embarassment and even ruin, both to manufacturers, merchants, and bankers, and great distress to the workmen.

[2] Lord King, in a letter to his tenants, declared that "In consequence of the late great depreciation of paper money, I can no longer consent to receive any bank notes at their nominal value in payment or satisfaction of an old contract." He required payments in guineas, or Portugal gold equal in weight to the number of guineas due, or in bank notes with an addition of £14. 12*s.* 6*d.* per cent., such being the difference in the market price of gold, when the agreements were made in 1807 and the market price in 1811. The government was reduced to the dilemma, either to strike at once sufficient gold coins, or to protect from arrest those who were unable to procure guineas to meet the demands on them. The latter was determined on, and an opportunity was lost of establishing a gold coinage of such a weight as would have secured it from disappearing.

On July 1, 1817, a new gold coin, called a sovereign of 20*s.* value, was made current by proclamation. It was 5 dwts. 3·274 grs. Troy weight of standard gold. On the obverse was the head of the king, with the circumscription GEORGIUS III. D. G. BRITANNIAR. REX F. D., and the date, and on the reverse, the image of St. George, armed on horseback, encountering the dragon with a spear, placed within the garter bearing the motto, HONI SOIT QUI MAL Y PENSE, with a graining on the edge (Stat. 57, Geo. III., c. 46). A new coinage of silver also was ordered. According to an account delivered to the House of Commons on 1 June, 1818, the coinage of sovereigns was 5,406,517, and of half-sovereigns 3,103,474, of shillings 50,490,000, and of sixpences 30,436,560.

George IV., 1820—1830. Various coinages were issued both for the united Kingdom and the Colonies during this reign.

In 1821, there was a silver coinage of crowns, shillings, and sixpences, and a gold coinage of sovereigns and half-sovereigns. The sovereign had on the reverse the figure of St. George and the Dragon, the half-sovereign, the ensigns armorial of the United Kingdom on a shield, surrounded by the rose, thistle, and shamrock, with the date.

Farthings were also struck, having on the obverse the king's head, with his titles; on the reverse, the figure of Britannia seated on a rock in the sea, holding a trident in her left hand and an olive branch in her right, with the inscription BRITANNIAR. REX. FID. DEF., and the date. In this year and the next, above 51 tons of copper were coined into farthings, 96 to the pound weight Avoirdupois.

In 1825, an Act was passed "to provide for the assimilation of the money and moneys of account throughout the United Kingdom of Great Britain and Ireland."

William IV., 1830—1837. William IV. ascended the throne on the death of his brother. A new coinage was ordered in 1830, of gold, silver, and copper, without any alteration of standard or weight, for the United Kingdom, and in the following years coinages were effected for several of the Colonies. By an order in Council, 3 Feb. 1836, a fourpenny piece in silver was struck for common circulation. The coin of this denomination was discontinued after the reign of Charles I., and was not struck for common currency until this year.

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[1] The following epigram appeared while the Bill for this Act was passing through Parliament:

Bank Notes and Guineas.

"Bank notes, it is said, once guineas defied
To swim to a point in trade's foaming tide;
But ere they could reach the opposite brink,
Bank notes cried to gold,—help us, cash us, we sink.
That paper should sink, and guineas should swim,
May appear to some folks a ridiculous whim;
But before they condemn, let them hear this suggestion,
In pun-making, gravity's out of the question."

On the reverse is the figure of Britannia, and on the obverse, FOURPENCE, with the date.

Victoria, 1837. Her Majesty Queen Victoria succeeded to the throne on the death of her uncle.

1838. An order in council directed the coinage of a sovereign and half sovereign of the same type, containing the ensigns armorial of the United Kingdom on a plain shield, surmounted by the royal crown, and encircled by a laurel wreath, with the inscription VICTORIA REGINA FID. DEF., with the rose, thistle, and shamrock placed beneath the shield. The type of the half-sovereign bears the arms without the wreath. The various coins struck during her Majesty's reign are in circulation, and require no description.

The following is a list of the names, weights, and current values of the coins in general circulation :

GOLD COINS.

The Sovereign ;	weight, 5 dwts. $3\frac{1}{4}$ grs. Troy ;	value, 20 shillings sterling
The half-sovereign ;	2 dwts. $13\frac{1}{2}$ grs.	10

The standard of the English gold coinage consists of 22 parts pure gold and 2 parts alloy. The word *carat* used as a weight is equal to $3\frac{1}{6}$ grains Troy, but when employed to express the fineness of gold, bears only a relative sense, as for instance, the English standard gold is said to be 22 carats fine, meaning, that it is composed of 22 parts pure gold and 2 parts alloy.

Gold coins being the standard of value, are the legal tender in payments to any amount whatever.

SILVER COINS.

The Crown	weight, 18 dwts. $4\frac{4}{11}$ grs. Troy,	value, 5 shillings
The Half-Crown	9 dwts. $2\frac{2}{11}$ grs.	2 shillings & 6 pence
The Florin	7 dwts. $6\frac{6}{11}$ grs.	2 shillings
The Shilling	3 dwts. $15\frac{3}{11}$ grs.	12 pence
The Sixpenny piece	1 dwt. $19\frac{7}{11}$ grs.	6 pence
The Fourpenny piece	1 dwt. $5\frac{1}{11}$ grs.	4 pence
The Threepenny piece	$21\frac{10}{11}$ grs.	3 pence

The standard silver consists of 11 oz. 2 dwts. of pure silver and 18 dwts. of alloy, or of 37 parts pure silver and 3 parts alloy; and was in the time of Edward I. called the old standard of England, which expression clearly proves that such must have been considered the standard for a long time before. Silver coins did not cease to be a legal tender before 1816, and are now only a legal tender for any sum not exceeding 40 shillings in one payment.

BRONZE COINS.

The Penny piece,	weight $\frac{1}{3}$ oz. Avoirdupois,	value 4 farthings
The Halfpenny piece,	$\frac{1}{5}$ oz	2 farthings
The farthing piece		1 farthing

The bronze-metal of the current coinage consists of 95 parts pure copper, 4 parts tin, and 1 part zinc, and coins were first issued for circulation in 1860.

EDITED BY

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OF WEIGHTS AND MEASURES.

BY ROBERT POTTS, M.A.,
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WEIGHTS AND MEASURES.

As the method of enumerating by tens was originally derived from the fingers of the human hand, so also in the same manner were the primary units of measures of length derived and named from the *members* of the human body, and from the *spaces* included in their ordinary motions. The primitive names of these measures in the languages of all nations prove the identity of their origin. Thus of the former are the lengths of the human foot, the nail, the finger-breadth, the hand-breadth or palm, and the cubit; and of the latter, the span, the step, the pace, and the fathom, employed as ordinary measures of length. The palm was reckoned the breadth of four fingers; the nail was the length from the end of the nail of the longest finger to the first joint; the inch, the length from the end of the thumb to its first joint, was adopted as an unit measure of length. This unit repeated four times was considered equal to the palm or the hand-breadth, and the hand-breadth taken three times gave the measure of the foot. The greatest expansion of the hand between the ends of the thumb and the middle finger gave the span; the distance from the end of the longest finger to the elbow, the cubit; the entire length of the arm, the yard; and the distance to which a man's two hands can be extended across the shoulders, the fathom. As the foot was too small an unit for estimating long distances with convenience, the step was reckoned equal to three feet; and the pace (*passus*), the interval between two steps, and equivalent to six feet, was also assumed as a measure; and a mile, as the word imports, consisted of one thousand paces (*mille passus*). Other measures of length have been derived and named from other considerations; as, for instance, the furlong (*furrow long*), taken to express the eighth part of a mile; the league (*lugen,* to see), a measure of three miles, supposed to express the distance the eye of a man, when standing upright, can see on a level plain; and the bow-shot, an ordinary measure of length formerly used in England and among other people who used the bow as a weapon. It will be evident that these measures, however sufficient they might be for the ordinary wants and conveniences of life in the early condition of human society, would be found to require more strict and exact definition as knowledge advanced, with the requirements of science and commerce.

The earliest lineal measures of which any certain knowledge has descended to modern times, are those of the Hebrews and Egyptians, the Greeks and the Romans. The most ancient writings of the Hebrews supply facts in evidence that Egypt in very early times was a country under a regular form of kingly government, its people civilised and trading with the people of other countries in the fruits and productions of those lands.

The oldest lineal measure named in these writings is the cubit, a measure used in describing the dimensions of the Ark (Gen. vi.)

In the description of the suburbs to the cities to be assigned to the Levites in Numb. xxxv. 4, 5, two cubits are referred to, one of which is double the length of the other. The cubit is stated to be the distance from the elbow bending inwards to the end of the middle finger, and called in Deut. iii. 11, "the cubit of a man." The larger cubit was called "the cubit of the armpit," as being measured from that part of the arm.

It may be presumed from the mention of the cubit in the later writings of the Hebrews, and in the writings of other nations, that the cubit was a measure known and generally recognised by other people besides those who came out of Egypt under Moses.

In the British Museum there is preserved a measure which was discovered at Karnak, on the removal of some stones, a few years ago, from one of the towers of a propylon, between which it appears to have been accidentally left by the masons at the time of its erection at the remote period of the eighteenth dynasty (about 1400 B.C.) It is divided into fourteen parts, but each part is double in length those of the elephantine, and therefore consists of four digits, and the whole measure is equal to 41·46 English inches. The double cubit has the first division in its scale of fourteen parts subdivided into halves, and the next into quarters, one of these last being equal to one digit. It is made of larch, and having been closed up in the building between two stones, excluded from the air, the wood is as sound as when it was used by the workmen. It is called the cubit of Karnak.

Herodotus (ii. 13) relates "that the priests told him that, in the reign of Mœris, whenever the waters of the Nile rose to the height of *eight cubits,* all the lands were overflowed, since which time 900 years have elapsed, and now (450 B.C.), unless the river rise *to sixteen,* or at least to fifteen cubits, its waters do not reach those lands." It is clear from this account that there was in Egypt a cubit in use, within less than a century after the Exodus of the Hebrews, double of the cubit used in Egypt in the age of Herodotus.

Among the artificers employed in the works of the Temple of Solomon, Hiram is described as the son of a widow of the tribe of Naphtali, whose territories were adjacent to those of the kingdom of Tyre, and his father was a man of Tyre, a worker in brass. Of this man it is recorded (1 Kings vii. 15), "He cast two pillars of brass 18 cubits high apiece;" but in 2 Chron. iii. 15, "He made before the house two pillars of 35 cubits high." If the words *casting* in the former, and *making* in the latter verse, have a difference of meaning, there is here no contradiction to the fact of one cubit being double the other.

By a comparison of the measures recorded in 2 Chron. iii. 3, and 1 Kings vi. 2, of the dimensions of the parts of Solomon's temple, with the account in Ezra vi. 3, of the rebuilding of the temple, it will appear that 60 cubits was the height of the latter, but 30 cubits the height of Solomon's, or that the cubit which describes the dimensions of the first temple was double of that which describes the second.

After the return of the Jews from Babylon at the end of the seventy years of captivity, Ezekiel states (ch. xliii. 13): "The cubit is a cubit and a hand-breadth," from which it would appear that the cubit there spoken of was a hand-breadth or a palm longer than another cubit.

Various opinions have been maintained of the length of those cubits.

Some maintain that the double Jewish cubit is the cubit of Karnak. Others that the cubit of five palms was the cubit used in the measures of the Temple of Solomon and the second Temple of Ezra, and called the royal cubit, and that the sacred cubit was that of six palms. This cubit is supposed to be of the greatest antiquity.

Sir Isaac Newton remarks in his dissertation on cubits, "that it is agreeable to suppose that the Jews, when they passed out of Chaldea, carried with them into Syria the cubit which they had received from their ancestors. This is confirmed both by the dimensions of Noah's Ark, preserved by tradition in this cubit, and by the agreement of this cubit with the two cubits which the Talmudists say were engraven on the sides of the city Susan during the empire of the Persians, and that one of them exceeded the sacred cubit half a digit, the other a whole digit. Susan was a city of Babylon, and, consequently, their cubits were Chaldean. We may conceive one of them to be the cubit of the royal city Susan, the other that of the city of Babylon. The sacred cubit, therefore, agreed with the cubits of divers provinces of Babylon, as far as they agreed with each other; and the difference was so small, that all of them might be derived, in different countries, from the same primitive cubits."

The conclusion of Sir Isaac Newton is, that the sacred cubit consists of 25·6 *unciæ* of the Roman foot. His reasons will be found in his "Dissertation on Cubits."

Dr. Hussey remarks:—"There is no certain method of obtaining an absolute value of any one element of the ancient Hebrew measures, from which a system of values might be calculated for the period before the captivity of the Jews. No weights, coins, nor measures of that age exist; and we must have recourse to probable inference or conjecture for determining the values of all."

The exact length of the Greek foot and of the Roman foot has engaged the attention of men of science in Europe for a very long period of time, and the result of their researches and investigations exhibit differences so small of these units in comparison with the foot measures of England and other countries, as to strengthen the presumption that all of them, having the same name, have had also the same origin.

Mr. Stuart measured the upper step of the basement of the front of the Parthenon at Athens, and found the length to be 101 feet 1·7 inches English measure. And if the name *hecatompedon* was applied to it on account of its length, he determined the length of the Greek foot from this measure to be 12·137 English inches.

In 1639 Mr. Greaves measured the Roman foot in the gardens of the Vatican,[1] and found it to be ·972 parts of the English foot, or

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[1] The following account is given by Professor Greaves himself in his works, vol. i., pp. 207–210:—

"In the year 1639 I went into Italy to view, as the other antiquities of the Romans, so especially those of weights and measures, and to take them with as much exactness as it was possible. I carried instruments with me made by the best artisans; where my first inquiry was after that monument of T. Statilius Vol. Aper, in the Vatican Gardens, from whence Philander took the dimensions of the Roman foot, as others have since borrowed it from him. In the copying out of this upon an English foot in brass, divided into 2000 parts, I spent at least two hours (which I mention to show with what diligence I proceeded in this and the rest), so often comparing the several divisions and digits of it respectively one with another, that I think more circumspection could not have been used, by which I plainly discovered

11·664 inches. He also found the foot of Cossutius to be ·967 parts of the English foot, or 11·604 inches. Several other measures of the Roman foot have been obtained, varying from one another, of which that of ·971 of the English foot has been generally received.

The Attic mina before Solon consisted of 72 drachmæ, or 6 in the ounce. They were possibly at that time only *laminæ argenti*, small and long pieces of silver, which, agreeably to the original meaning of the word drachma, might be grasped in the hand. But whatever might have been the form of the Athenian money before Solon, the mina or pound of 72 drachmæ, by his celebrated Σεισάχθεια, was altered to 100. In other words, Solon introduced a nominal pound instead of a real one; for the Athenian pound had no more than 84 drachmæ coined out of it for many ages afterwards.

The Romans from the beginning of their coining silver accommodated themselves to the Grecian practice; their pound of silver was made into 84 denarii, but in tale it was always 100.

It is probable that the different States of Greece had in use different weights and measures; for on the rise of the Achæan League, B.C. 280, it was agreed that the twelve associate cities should not only be governed by the same laws, but should be bound to use the same money, weights, and measures (Polyb. 129). It is not to be expected that the weights and measures of ancient times would be made with the same exactness as in modern times, as the physical facts, by means of which extreme accuracy can be attained, were either unknown or not taken into account.

The Roman pound of Byzantium, preserved in the British Museum, weighs 4995 grains Troy; and the pound of gold was made into 72 coins, nominally each of 69·4 grains, but really of 68 grains.

The Greek and Roman physicians above all others are likely to be the best judges of the relative proportions of the weights and measures used in their prescriptions. The Greek physicians, being in the highest repute at Rome, adopted as a weight the *drachma* instead of the *denarius*, as the latter was not long after applied to coins of a lower class, and lost its former estimate in weight. Scribonius Largus, a physician, who attended the Emperor Claudius into Britain, and recorded some circumstances in that voyage, "Cum Britanniam peteremus cum Deo nostro Cæsare," represents it indifferent whether prescriptions are made by the denarius or the drachma. And Galen has observed that in his time the denarius and drachma were but different names for the same weights.

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the rudeness and insufficiency of that foot. For besides that the length of it is somewhat too much (whatsoever Latinius, out of an observation made by Ant. Augustinus, Sighicellus, Pacatus, Maffeus, Statius, Ægius, and Fulvius Ursinus, pretends to the contrary) there is never a digit that is precisely answerable to one another. Howsoever, it contains 1944 such parts as the English foot contains 2000.

"My next search was for the foot on the monument of Cossutius, *in hortis Colotianis*, from whence it has received its denomination (though it be now removed), being termed by writers Pes Colotianus. This foot I took with great care, as it did well deserve, being very fair and perfect; afterwards collating it with that Roman foot which Lucas Pætus caused to be engraven in the Capitol on a white marble stone. I found them exactly to agree; and therefore I did wonder why he should condemn this with his pen (for he makes some objections against it), which, notwithstanding, he hath erected with his hands, as appears by the inscription in the Capitol, CURANTE LU. PÆTO. It may be, upon second thoughts, he afterwards privately retracted his error, which he was not willing to publish to the world. Now this foot of Cossutius is 1934 such parts as the English foot containing 2000."

Galen discovered, by comparing the measures and the scale weights, that the horn used for measuring oil at Rome held one-sixth part less than the weight of the libra. The metrical horn was divided into twelve parts or ounces, like the libra; and each of these divisions was commonly supposed to be equal to an ounce scale weight, and passed for such in estimation. Galen, as a physician, thought it requisite to interpose some caution; that, in prescribing oil, they ought to distinguish by what ounces they prescribed, whether the scale ounce or the metrical ounce: as the scale pound determined the weight of bodies, but the metrical only the contents, or the quantity of space they filled.

The pound of 100 denarii or 100 drachmæ was merely nominal, and employed for the purpose of trade. Both the denarius and the drachma, during the Consular and part of the Imperial Government, were only seven to the ounce. They began to lessen under Nero. What Pliny writes of the aurei is equally true of the silver coins, "Paullatim principes imminuere pondus." They probably did not coin 8 in the ounce till the time of Galen, about a century after Nero. A drachma has ever since been reckoned the eighth part of the ounce, and probably was so some time before. Thus the drachma reckoned at 6 to the ounce, gives 72 in the pound; at 7, gives 84; and at 8, 96 in the pound.

The congius of Vespasian supplies a method of discovering the true weight of the Roman pound. This congius was made A.D. 75, according to the standard of measure preserved in the Capitol, and contains ten pounds, as the inscription upon it testifies: "IMP. CÆSARE VI. T. CÆS. AUG. F. IIII. COS. MENSURÆ EXACTÆ IN CAPITOLIO. P.X." It is commonly called the Farnese Congius, and is now preserved at Dresden. In 1824 Dr. Hase had it filled with distilled water and carefully weighed, and the weight of the water was found to be equal to 52037·69 grains Troy weight, giving 5203·769, or nearly 5204 grains to the Roman pound.

It has been remarked by Dr. Hussey (Essay on Ancient Weights) that "the ancient metallic weights in themselves are too unequal and inconsistent with each other to give any certain result." He regards more favourably the two obtained from the excavations at Herculaneum—the only two large specimens which are perfect—one of 50 and the other of 100 Roman pounds. Of these, the former weighing 256564 grains Troy, gives a pound of 5131·28 grains; the latter, weighing 518364 grains, gives a pound of 5183·64 grains, which is between 20 and 21 grains less than that given by the congius of Vespasian. If the experiment with the congius had been made with water not distilled, the result would have shown a less difference. And so small a difference in the two calculations is no matter of surprise, when much greater differences are found between the old Roman weights themselves. The modern Roman pound contains 5236 grains Troy, and exceeds the old Roman pound, as given by the congius, by only 32 grains. This fact affords a strong presumption that the ancient and modern weights were identical, and that the difference is not greater than was likely to be found in the lapse of so many centuries.

The Roman pound *libra*, called also *as*, was divided into twelve equal parts, called unciæ (ounces); and for every number of ounces under 12, the Latin language has a distinct name.

The Anglo-Saxon weights and measures were established throughout the whole of England. The weight of the Saxon pound was not likely to be changed, as the same pound and the same division of the pound prevailed at that time over the greater part of Europe. This is certain of the common pound of Italy as early as A.D. 958, if not much earlier; for one of their records makes mention of an estate sold in that country for 60 pounds of 240 pennies to the pound. In the time of King Alfred the Saxon pound weight was 240 pennies, and was estimated by the mancus, making 8 to the pound, each of 30 pennies; and also by 48 shillings, each of 5 pennies. The mancus (*manu cusum*) was probably derived from Italy, in the intercourse between that country and England after the friendly reception by Ethelbert of the mission of Augustine in 597. It appears the mancus was a name applied to a weight. Archbishop Ælfric, at the end of the tenth century, states that the mancus was equal to 30 pennies or 6 shillings.

It is recorded that King Edgar, 975 A.D., with the consent of his council, decreed "that one and the same money should be current throughout his dominions, which no man must refuse; and that the measure of Winchester should be the standard, and that a weigh of wool should be sold for half a pound of money, and no more."

The Saxon coins were regulated by the pound weight brought from Germany, and afterwards known by the name of the Cologne pound weight.[1] The precise weight of the Saxon money pound cannot be exactly ascertained from positive evidence, but there is strong presumptive evidence, first shown by Mr. Foulkes ("Tables of Silver Coins," p. 3, note), that it was of the same weight as that known for some centuries after the Conquest by the name of the Tower pound, and was so named from the fact of the principal mint of England being in the Tower of London. The Saxon pound was, like the Tower pound, divided into 12 ounces. If the supposition of the identity of the Saxon and Tower pound be correct, the Saxon pound contained

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[1] Mr. Clarke has given in his "Connection of Roman, Saxon, and English Coins," p. 24, the following weights of the ounce :—

	Grains, Troy.
The Strasburgh ounce from standards made 1238 A.D.	451·38
The present Strasburgh ounce	454·75
The old Saxon ounce, from the Chamber of Accounts in Paris, about the time of Edward III., after 1327, A.D.	451·76
The present Cologne ounce	451·38
The old Tower ounce, as taken from the accounts in the English Exchequer, 1527, A.D.	450·00

The small differences of these several ounces seem to suggest that all of them had the same origin. The immemorial usage of the Cologne or Strasburgh pound in Germany, and in Britain from the first arrival of the Saxons till the time of Henry VII., is evidence of its great antiquity.

Dr. Arbuthnot makes the Greek ounce to consist of 455·33 grains Troy, which is nearly identical to the present Strasburgh ounce. The Saxon writers refer the origin of their nation to the Getæ, to whom the Goths and Germans were related as kindred clans. The near agreement of the Greek ounce and the German ounce may form a ground for the presumption that the Greeks and the Germans may have descended through different branches from the same people. This presumption derives some probability from the similarity of usage in the Greek and Saxon languages of the article, the double negative, and the formation of proper names.

Ovid, in some pieces written during his exile in Pontus, noted an affinity between the Greek and and Getic tongues, and remarked that though the Getic tongue was disguised by a barbarous pronunciation, there were evident marks of its Greek original.—*Ovid, Trist.*, v. 7, 10.

5400 grains Troy, and this weight of silver coins was a pound sterling. The pound of silver was always reckoned by account at something more than the number of pennies which were struck out of it at the mint. It was a Roman as well as a Saxon custom.[1] When the Romans coined 8 denarii to the ounce, and 96 to their pound of silver, they paid 100 in tale; and when the Saxons coined 4 shillings to the ounce, and 48 to their pound, they paid 50 in account. And it may be noted that the Saxon laws always reckoned their pound in the round number of 50 shillings, when it is evident they really coined out of it only 48. Mr. Foulkes's discovery of the old Tower pound being the same as the Anglo-Saxon pound of the moneyers, has made out the pound of 48 shillings.

When William the Conqueror had ratified the laws of Edward the Confessor, he decreed that the weights and measures which had been established by his predecessors should be continued in use, and ordered that the measures and weights should be true, and stamped in all parts of the kingdom.[2] The same regulations were continued by his successors. Richard I., who ascended the throne in 1189, not only enforced an uniformity of measures, but also ordered that the vessels employed for measuring should be edged with hoops of iron, and that standards should be kept by the magistrates in the different counties and towns of the kingdom. And this uniformity of weights and measures was confirmed by the Magna Charta in 1225. The Act 5 Henry III., c. 9, ordained that "One measure of wine shall be throughout our Realm, and one measure of ale, and one measure of corn, that is to say, the quarter of London; and one breadth of dyed cloth, that is to say, two yards within the lists. And it shall be of weights as it is of measures." And by the statute of 51 Henry III., 1266, they were defined in the following form:—"By the consent of the whole Realm of England, the measure of our Lord the King was made; that is to say, that an English peny, called a sterling, round and without any clipping, shall weigh 32 wheat corns in the midst of the ear,[3] and 20 pennies do make an ounce, and 12 ounces one pound, and 8 pounds do make a gallon of wine, and 8 gallons of wine do make a London bushel, which is the eighth part of a quarter."[1] The bushel here named was called the Winchester measure.

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[1] In the times of the later Roman emperors, there appears to have been a custom of making allowances in the payment of money by tale. This is evident from a law of the elder Valentinian, by which he enforced a law of Constantine for paying by weight; and he observes that it was usual to take two or three solidi in the pound as the common allowance in tale. "Facile enim eos provinciæ rector a dispendio vindicabit, qui binis ceu ternis solidis necessitudinem solutionis adimpleverit." It may be added, that both during the Republic and under some of the emperors, the Romans coined 84 denarii out of a pound weight of silver, though the pound by tale was always reckoned at 100.

[2] On the accession of William I. to the throne of England, the pound in tale of the silver coins current was equal to the pound weight of standard silver, that is, the moneyer's pound, afterwards called the Tower pound. The *pound in tale* was divided into twenty shillings, and each shilling into twelve pennies, or sterlings. The *pound in weight* was divided into twelve ounces, and each ounce into twenty pennyweights, so that each penny weighed one pennyweight. The only coins made at this period were pennies. This simple system of coinage, by which the pound in tale was made equal to the pound in weight, and was divided in the manner before mentioned, is supposed to have been first introduced into France towards the end of the eighth century. As the Norman princes for a long time before had considerable intercourse with England, it may have been introduced from France in the times of our Anglo-Saxon ancestors. This system of silver coinage continued without any alteration in the weight of the silver coins until the year 28 Edward I.

[3] The grain of barley and the seed of the *Abrus precatorius* were assumed by the

There is a twofold relation stated between *weight* and *volume*, for

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Hindus as the primary elements of their weights. The weight of the seed of the abrus was taken equal to two barleycorns.

Mr. Thomas has shown that, from comparative numismatic data of various ages, he has found the true weight of the rati or gunja seed to be 1$\frac{3}{4}$ grains Troy. In the sixth volume of the Royal Asiatic Society he makes the following remarks, pp. 342, 343 :—"The determination of the true weight of the *rati* has done much both to facilitate and give authority to the comparison of the ultimately divergent standards of the Ethnic kingdoms of India. Having discovered the guiding *unit*, all other calculations become simple, and present singularly convincing results, notwithstanding the basis of all these estimates rests upon so erratic a test as the growth of the seed of the gunja creeper (*Abrus precatorius*), under the varied incidents of soil and climate. Nevertheless this small compact grain, checked in early times by other products of nature, is seen to have had the remarkable faculty of securing a uniform average throughout the entire continent of India, which only came to be disturbed when monarchs, like Shir Shah and Akbar, in their vanity, raised the weight of the coinage without any reference to the number of *ratis* inherited from Hindu sources as the *given* standard, officially recognised in the old, but altogether disregarded and left undefined in the reformed Muhammedan mintages."—*Journal of the Royal Asiatic Society, Vol.* vi., *p.* 342. *Article by Edward Thomas, F.R.S.*

"The carat is a bean, the fruit of an Abyssinian tree called kuara. This bean from the time of its being gathered varies very little in its weight, and seems to have been, in the earliest ages, a weight for gold in Africa."—*Bruce's Travels*, v., p. 66.

The carat has been adopted by most European nations in estimating the weight and purity of the precious metals. As used by goldsmiths the carat is a weight for gold. They divide the ounce Troy into twenty-four parts named carats, and each carat into four grains, so that gold twenty-two carats fine means that an ounce of standard gold contains twenty-two parts pure gold and two parts alloy.

[1] In the largest chamber of the Great Pyramid of Gizeh (Herod. ii. 124) is a rectangular vessel cut out of a single block of Theban marble or porphyry. This vessel is known by the name of the Pyramid Coffer, or the Porphyry Coffer. A singular coincidence has been found to exist between the capacity of this coffer and four quarters English measure. This sameness of volume affords a strong presumption that some relation exists between the measure of capacity of this ancient vessel and the measure of four quarters, or a chaldron. This vessel has, at different times, been carefully measured by scientific men, three of whom have reported the following dimensions :—

Professor Greaves in 1638–9 visited the pyramid and very carefully took the interior dimensions of the coffer, and found them to be—Length, 77·856 inches; breadth, 26·616 inches; depth, 34·320 inches; giving the content 71118·4 cubic inches.

M. Jomard in 1799 reported the dimensions—Length, 77·806 inches; breadth, 26·599 inches; depth, 34·298 inches; giving the content 70982·4 cubic inches.

Colonel Howard Vyse took the measures in 1837 and found them to be—Length, 78·0 inches; breadth, 26·5 inches; depth, 34·5 inches; giving the content 71311 cubic inches.

The English gallon contains 277·274 cubic inches, and the measure of corn called "*the quarter*" will contain 17745·536 cubic inches, and consequently *the unit measure* of which this is the quarter will contain 70982·144 cubic inches. Hence it appears that four English quarters have very nearly the same volume as the pyramid coffer according to the measurement of M. Jomard. This identity of volume is surprising, and can scarcely be regarded as accidental. It may be observed that it is highly probable that the coffer of the pyramid has been in the chamber for a period above 4,000 years.

The student may read some interesting information in the following works :—Pyramidographia, or a Description of the Pyramids in Egypt. By John Greaves, Professor of Astronomy in the University of Oxford. London, 1646. The Great Pyramid. Why was it built, and who built it? By John Taylor, 1859. Our Inheritance in the Great Pyramid. By Prof. C. Piazzi Smyth, F.R.S.S. L. and E., Astronomer Royal for Scotland. London, 1864.

testing wheat and wine. A measure which would contain as much wheat as would be equal in weight to eight gallons of wine, must contain ten gallons, or be one-fourth larger than the wine measure. The London bushel containing 64 pounds of wheat would hold 80 pounds of water. The law of only one measure for wheat and for ale will require that eight gallons of ale must weigh one-fourth more than eight gallons of wheat.

In the year 1306 a statute of 33 Edward I. ordains that a foot shall contain 12 inches, and an inch 3 barley-corns[1] placed end to end, taken out of the midst of the ear of barley. By another ordinance for the measure of land, the acre is also declared to contain 160 square perches, or 16 in length and 10 in breadth, and another statute names other lengths and breadths of an acre.

In the year 1324, the 17 Edward II., the statute of Edward I. was more fully described, in the ordinance for the determination of measures of length and surface. It was ordained that three barley-corns round and dry, make an inch, twelve inches a foot, three feet a yard, five yards and a-half a perch, and forty perches in length and four in breadth an acre.

14 Edw. III., 1340, in pursuance of the intent of Magna Charta, the King's treasurer was directed to cause standards to be made and sent into all the counties which were not already provided with them. And in 1350 an order was issued recognising the laws of the Great Charter, and declaring that every measure of corn shall be stricken without heap.

If the measure of wheat was *struck off* level with the rim of the measure, the weight became ruled by the measure of capacity; whereas before that time the measure of capacity was governed by the weight. Hence, as the *strike* bushel contained only 62 pounds, while the old heaped bushel contained 64 pounds, this reduction of the bushel would reduce the gallon from 280 to 272 cubic inches. And the wine gallon was increased from 224 to 231 cubic inches.

By the statute 25 Edw. III., on account of great damage and deceit done to the people by a weight called "*auncel*," an order was issued that auncel weight should not be used, but the weights used should be according to the standard of the Exchequer, and that the beam of the balance bow not more to one part than to the other. And two years after, another statute (after noting that some merchants do buy, "*avoir de pois*,"[2] wools and other merchandises by one weight and sell by another) ordains that one weight, and one measure, and one yard, be through all the land, and that wools and all manner *avoir de pois* be weighed by the balance, so that the tongue of the balance be even.

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[1] It is a curious coincidence to find in the Lilavati, the barley-corn employed as a primary unit of length. "Eight breadths of a barleycorn, or three grains of rice in length, make a finger; four times six fingers a forearm or cubit; and other measures of length are formed from these." The Italians, who received their knowledge of the Indian arithmetic from the Arabians, take the origin of their measures from the barley-corn (*grano di orgio*), and take four to make a dedo (*digitus*) or finger; four dedi a palmo, &c.

[2] The first notice of Avoirdupois with merchandise in the English laws, occurs in 1335, the 9 Edwd. III., 1 stat., c. 1:—". . . . ordine est et establi qe touz marchantz alienz et denzeins et touz autres et chescuns de eux de quel estat ou condition qils soient qi achatre ou vendre voillent blez, vins *avoir de pois* chares pesson et touz autres vivres et vitailles laines drapz mercez marchandises et tote manere dautres choses vendables. . . . ." In other statutes, the expression is written *aver du pois* and *avoir du pois*. The expression avoir du pois may have reference to the weight by which the mercantile pound exceeded the moneyers' pound, and was never employed in weighing the precious metals, but only in estimating the weight of coarse and heavy wares and articles of merchandise. In the Latin Commentary on the

In the year 1389, the 13th Rich. II., uniformity of weights and measures was ordained throughout the kingdom, "except in the County of Lancaster, because in that county it hath always been used to have greater measure than in any other part of the realm."

An ordinance of 15th Rich. II. directed that 8 bushels stricken be the quarter of corn, and penalties be enforced on all who made use of 9 bushels to the quarter.

By statute of 2 Hen. VI., 1424, a tun of wine was defined to contain 252 gallons; a pipe, 126 gallons; a tertian, 84 gallons; and a hogshead, 63 gallons. At a later period the measure called a tertian or tierce, being the third part of a pipe or butt, was called a puncheon. Another statute of 1429 required all cities, boroughs, and towns to procure a common balance and weights at their own expense, according to the standard of the Exchequer, to be kept by the mayor or constable.

In the year 1495, on petition of the Commons, an Act, 11 Hen. VII., was passed, that weights and measures according to the standard of the Exchequer should be delivered for safe custody to the mayors of boroughs and corporate towns; that all common weights and measures be made according to these standards, and marked as the king's standard; and that all weights and measures in use should be examined twice every year. It was also ordained "that eight bushels raised and stricken shall be a quarter of corn; and fourteen pounds a stone of wool; and twenty-six stone a sack." On more diligent examination some of the measures approved were found deficient, not having been made according to the old laws of the realm.

By an amended statute, 12 Hen. VII., c. 5, it was enacted "that the measure of a bushel contain eight gallons, and that every gallon contain eight pounds of wheat Troy weight, and every pound contain twelve ounces of Troy weight, and every ounce contain twenty sterlings, and every sterling be of the weight of thirty-two corns of wheat that grew in the midst of the ear of wheat, according to the old laws of the land." The Tower pound then in use, and considered the same as the old Saxon pound, was, by this Act, exchanged for the pound Troy. This alteration was owing to the *Intercursus Magnus*, or Great Treaty of Commerce concluded between England and

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English Laws, entitled *Fleta*, composed in the time of Edward I. (Lib. iii., c. 12), it is stated expressly that fifteen ounces make the merchant's pound. The Saxon practice was the same as the Roman. They both had two pounds, one for the Exchequer and one for trade, having the ounces the same in both, but the commercial pound differing from the other only in the number of ounces. This difference between the Saxon mercantile pound of 15 ounces and the Roman pound of 16 ounces was not considerable, not more than half an ounce, and they might pass mutually for each other in trade without much inconvenience. The account of Fleta confirms the fact that the Conqueror's laws confirmed the Saxon practice, and that the subsequent kings followed the same rule, and he also states that the usage continued to his time, and most probably till the reign of Edward III. For in the ninth year of his reign, the prohibition on foreign merchants was removed, and they were at liberty to buy all Avoirdupois wares and merchandise at any place without the realm, and to sell them to any persons whatsoever except the king's enemies. It appears from this Act, and from several others, that the commerce of England was far more considerable in those early times than is commonly imagined.

Flanders the year before. The Flemish pound was adopted as a compliment to the Duchess of Burgundy, and for the mutual convenience of all payments in future, which would be adjusted according to this pound.

The following proclamation of 18 Hen. VIII., November 5, was issued confirming the stat. 12 Hen. VII. c. 5:—"And whereas heretofore every person who brought bullion to the king's mint to be coined paid two shillings and sixpence for the coinage of every pound Tower weight, which differed from the pound Troy three-fourths of an ounce in the pound weight, it was determined that the pound Tower should no more be used, but that all gold and silver should be weighed by the pound Troy, being of twelve ounces, and heavier than the Tower pound by three-fourths of an ounce."

In 1570, the 13th year of Elizabeth, an Act declared 28 gallons of old standard to be about 32 gallons wine measure; whence the old gallon must have contained 264 cubic inches, or a little less than the Winchester gallon, if the wine gallon contained 231; or rather, since the standard gallon of Elizabeth in the Exchequer actually contained 271 cubic inches, the wine gallon of that day must have contained 237.

By a statute of William III., the Winchester bushel is declared to be round, with a plane bottom 18½ inches wide throughout, and 8 inches in depth; hence its content must be 2150·42 cubic inches, and a gallon dry measure 268·8 cubic inches; while by an Act of the next reign a wine gallon is declared to be a cylinder, 7 inches in diameter and 6 inches deep, or 231 cubic inches.

In the year 1758 was appointed a Committee on weights and measures, and they reported that serious discrepancies existed in the measure called the gallon, and that they found three standard gallon measures differing from one another. Mr. Bird was authorised to ascertain the exact contents, in cubic inches of water, of each of the standard measures in the Exchequer. He was also ordered to make two brass rods according to the standard yard of the Exchequer, and these, on careful comparison with the yard of the Royal Society, were found to agree as nearly as possible. The Committee recommended the one which the legislature might approve to be marked "Standard Yard, 1758." Though these standards formed by Mr. Bird were not made legal, as the Bill of 1760 for that purpose did not pass into a law, they became the foundation of all the measures which have since been used in England.

Another report was made in the following year, in which it was recited that ever since Magna Charta one weight and one measure was ordained to be used throughout the realm, but that the means adopted in succeeding reigns from Edward III. to Henry VII. had not proved effectual for that purpose.

It will be seen that measures were sometimes to be estimated by weights, or weights by measures; and the standards having been determined, they were in practice still liable to considerable modification, according to the manner in which they were employed, and the nature of the substances concerned, so that various directions for weighing and measuring have been given in the statutes of different reigns, and different allowances have been made for different articles.

In measures of length, the custom of interposing the thumb when cloth was measured by the yard had been so universal that the thumb came to be considered as a part of the measure, and in process of time an inch was substituted for it, so that a yard was made to consist of 37 inches. This was recognised in 1404, in the reign of Henry IV., when it was ordered that the yard was to have an inch added to it, containing the breadth of a man's thumb.

An Act of 10 Anne, in 1711, directed that each yard of cloth should in measurement have one inch added to it "instead of that which is commonly called a thumb's breadth." There were besides different regulations ordered in different counties for different kinds of cloth, both of linen and of wool. Nothing is prescribed by the laws respecting any particular temperature at which measures are to be employed, the effect of any difference being too inconsiderable to be perceived in any common case; but the state of dryness and moisture is by no means indifferent to the result, and particular directions have been given in many statutes respecting the effect of wetting cloth on the measure of its length and breadth.

In 1790, the Constituent Assembly of France, on the proposition of M. Talleyrand, agreed to invite the British Government to concur with the French nation on fixing a natural unit of measures founded on the length of the seconds pendulum in the latitude of 45°. A Commission of Members of the Academy of Sciences was appointed, and their report appeared in the following year.

Delambre and his colleagues, after much interruption, completed their operations in 1796. A Commission of Members of the Institute was appointed to inspect and revise the recorded observations and calculations, as well as the instruments employed in the course of the operations. The Commission determined the length[1] of the mètre from the quadrant of the meridian to be equal to 443·295936 lignes, being less than the mètre provisoire (which was founded on La Caille's measure of a degree of the meridian in the latitude of forty-five degrees) by ·146 of a ligne.

The subsequent determination of the unit of weight, the kilogramme or decimètre cubed of distilled water, was made at its greatest density, and not, as it was at first proposed, at the temperature of melting snow.

On 19 Aug. 1798, the original mètre and kilogramme were presented, with a pompous address, to the two Councils of the Legislative Body of France: "This unit" (the mètre), say they, "offers also one aspect which is not without interest. It must be pleasing to the father of a family to say, 'The field which supports my children is such a portion of the globe. I am in this proportion co-proprietor of the world.'" And further noticing that these prototypes shall be deposited among the national archives, to be preserved with religious care; and that "the ignorance and ferocity of barbarians shall never bear them away as trophies from the valour, the patriotism, and the virtues of a nation enlightened in its interests, its honour, and its rights. But if an earthquake should swallow them up, or if it were possible that a terrific blast of lightning should fuse the metal which is the conservator of this measure, it will not therefore follow, citizen legislators, that the fruit of so many labours, that the general type of our measures, shall be lost to the national glory and the public benefit."

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[1] They proposed that the ten-millionth part of the quadrant of the meridian should be called the *mètre*, and be considered as the *unit* of the new metrical system of France.

That in order to determine the mètre, an arc of the meridian extending from Dunkirk to Barcelona, six degrees and a half to the north, and three degrees to the south of the mean parallel of forty-five degrees, should be measured.

That the quadrant should be divided into one hundred degrees, the degree into one hundred minutes, and the minute into one hundred seconds.

That the weight of a decimètre cubed of distilled water at the temperature of melting ice should be determined as the unit of measures of weight.

That the subdivisions of all measures should be adapted to the decimal scale.

On the grand question of an international system of weights and measures, the British Government at that time appears not to have appreciated the lofty aims of the Constituent Assembly of France; notwithstanding its professions of being "so renowned for its feelings as to the government of nations and the general welfare of mankind." A motion, however, was made on the question in the House of Commons, but no further steps appear to have followed the motion.

It was not before the year 1837 that the metric system was made legal in France, but then so as not to be brought into use before 1840. It has been affirmed that the adoption of the system had been postponed chiefly from political motives, and by the hatred which attached to everything connected with the Revolution. In some districts in France at the present day, and even in Paris, the ancient measures have not yet been wholly superseded.

In 1798 an account of some endeavours to ascertain a standard of weight and measure by Sir George Shuckburgh Evelyn was printed in the Transactions of the Royal Society. He states that in 1780 he had taken up the idea of an universal measure from whence all the rest might be derived by means of a pendulum with a moveable centre of suspension. It appears that Mr. Whitehurst, F.R.S., published a pamphlet in 1787, in which he described a method of finding an invariable length. The mechanism he employed consisted of a pendulum of variable length, which was kept vibrating by means of clockwork. The standard measure of length he defined to be nothing more than the difference of the lengths of two pendulums, which vibrate in different but ascertained times. Sir G. S. Evelyn obtained the use of this machine, and having ascertained that the difference between the pendulum which vibrates 42 times, and that which vibrates 84 times in a minute, is equal to 59·89358 English inches, he made use of that length for the determination of a standard of length.

To ascertain a standard of weight he provided microscopes and micrometers for the most exact observations; a hydrostatic balance, which when loaded with six pounds had its equilibrium disturbed by the hundredth part of a grain; a solid cube of brass, whose edge was five inches; and a solid brass cylinder, whose length was six inches and diameter of the base four inches. He employed pure distilled water, and weighing the cube in air and in water he found the weight of a volume of water equal to the volume of the brass cube. The same operation was performed with the brass cylinder, and on comparing the results of these two, and by other experiments, he determined the weight of a certain volume of distilled water to a very great degree of accuracy.

The two results arrived at by these experiments may be thus briefly expressed:—The difference of the lengths of the two pendulums vibrating 42 and 84 times in a minute of mean time, in the latitude of London, at 113 feet above the level of the sea, at 60° of

Fahrenheit's thermometer, the barometer being at 30. inches, is 59·89358 inches of the Parliamentary standard. And that according to the same scale of inches, a cubic inch of pure distilled water, at 66° of Fahrenheit, when the barometer is at 29·74 inches, weighs 252·422 Parliamentary grains.

In 1814 a Select Committee was appointed under the Privy Seal, to inquire into the original standards of weights and measures in the kingdom, and into the laws relating thereto. They state in their report that the Statute Book from the time of Henry III. abounds with laws for maintaining an uniformity of weights and measures throughout the realm. They remark that the measures prescribed in the Exchequer are at variance with one another, and, as the law stands, all of them are considered to contain like quantities; that the laws are at variance with one another, and that exceptions are allowed in different counties, as well as in some articles of merchandise. They attribute the chief causes of the existing inaccuracies to the want of a fixed standard in nature, and a simple mode of connecting measures of length with those of capacity and weight. In order to compare the standard of length with some invariable natural standard, they reported that it appears from the experiments made for determining the length of a pendulum vibrating seconds in the latitude of London in a vacuum, and reduced to the level of the sea, that the distance from the axis of suspension to the centre of oscillation of such pendulum is 39·1393 inches, of which the standard yard contains 36. Next, that the weight of water contained in any vessel affords the best measure of its capacity. If the standard gallon of water weigh 10 pounds avoirdupois and contain 276·48 cubic inches, the quart will weigh 40 ounces and contain 69·12 cubic inches, and the pint will weigh 20 ounces and contain 34·56 cubic inches. And such a simple connection between the standards of weight and measure of capacity will most likely tend to preserve from error the uniformity of these measures. And in this manner the standard of length is kept invariable by means of the pendulum, the standard of weight by the standard of length, and the standard of the measure of capacity by that of weight. The Committee concluded their report with twenty resolutions for the consideration of the legislature.

It does not appear that these recommendations were so far approved by the House as to enter upon legislation; but in 1818 a Commission was appointed further to consider the subject of weights and measures, and the Commissioners made their first report in 1819. In the appendix to their report they make the following remarks:—

"A general uniformity of weights and measures is so obviously desirable in every commercial country, in order to the saving of time, the preventing of mistakes, and the avoiding of litigation, that its establishment has been a fundamental principle in the English Constitution from time immemorial; and it has occasionally been enforced by penal statutes, and by various other legislative enactments. At the same time, it has commonly been considered as one of those objects which cannot, consistently with logical accuracy, with natural justice, and with the liberty of the subject, be very precisely defined, or very peremptorily and arbitrarily enjoined on every occasion; and there are many instances in which a departure from complete uniformity is not only tolerated, but established by law. It must, indeed, sometimes be almost as impossible to control the despotic influence of custom with respect to the contents of a measure of a certain denomination as with respect to the signification of a word of any other nature; and even the terms of numbers, precise as they necessarily are in their strict meaning, have become liable to perpetual variations, according to the objects to which they are applied; and these variations, however inconvenient they may appear upon a general view of the subject, have been repeatedly sanctioned by their adoption in the Acts of the legislature." The Commissioners also observe, "With respect to the actual magnitude of the standards of length, there did not appear to be any sufficient reason for altering those which were in general use." They further remark that "there is no practical advantage in having a quantity commensurable to any original quantity existing, or which may be imagined to exist in nature, except as affording some little encouragement to its common adoption by neighbouring nations. But it is scarcely possible *that the departure from a standard once universally established in a great country, should not produce much more labour and inconvenience in its internal relations, than ever it could even be expected to save in the operations of foreign commerce and correspondence.* The subdivision of weights and measures, at present employed in this country, appears to be far more convenient for practical purposes than the decimal scale, which might perhaps be performed by some persons for making calculations with quantities already determined. But the power of expressing a third, a fourth, and a sixth of a foot in inches, without a fraction, is a peculiar advantage in the duodecimal scale; and for the operations of weighing and measuring capacities, the continual division by two renders it practicable to make up any given quantity with the smallest number of standard weights and measures, and is far preferable in this respect to any decimal scale."

In the appendix to the second report, made in 1820, is an account of the variations of weights and measures then in use in different parts of the kingdom. A few of them are noted. The pound Avoirdupois of 16 ounces was found in some places to be reckoned at 24 ounces. It is noted that in the time of 31 Edward I., the stone was reckoned to be one-eighth of 100 pounds, or 12½ pounds. In later times the stone was reckoned at 8 pounds in some places, in others 14 or 16. A hundredweight was originally 100 pounds, but the meaning of the word hundred had varied in its use. At one time it meant 108 pounds, and at length became fixed to 112.

In reckoning articles of merchandise, a score numerically meant twenty articles, so that 5 score made 100. In some articles of commerce 6 score is reckoned to the hundred, which has been named "a long hundred." And "a great hundred" contained 24 long hundreds of 120 each. The word dozen was the same as 12, but in some articles of commerce 13 were taken to the dozen. Twelve dozen was called the small gross, and the great gross consisted of 12 small gross.

The oldest measure for wine is a gallon preserved in the Guildhall, London, and said to contain 224 cubic inches; in 1818 its exact capacity was 224·4 cubic inches. The Exchequer wine gallon, dated 1707, was found to contain 133·4 ounces, answering to 230·9 cubic inches; while the measurement of 1758 made it 231·2 cubic inches. A duplicate of this measure, and of the same date, was preserved at the Guildhall, London.

The ale gallon contained 4½ per cent. more than the corn gallon.

The Winchester gallon, according to the definition in the statute 1 William and Mary, should contain 269 cubic inches, while in other Acts it was fixed at 272½ cubic inches. The ale gallon of the Exchequer contained 282 cubic inches, while the wine gallon was fixed by statute, 5 Anne, at 231 cubic inches. One of the standard pints of the Exchequer was found to contain 20 ounces of distilled water, which most probably suggested the assumption that the imperial gallon should contain ten pounds weight of distilled water, being exactly eight times the weight of the pint of the Exchequer.

The third report concisely refers to the results obtained, and recommends the adoption of the regulations and modifications suggested in the two former reports.

In the year 1820, the first year of the reign of George IV., a select committee was appointed to consider the reports laid before the House relating to weights and measures. In their report of the date of 28th May, 1821, they state that "They concur entirely in opinion with the Commissioners on Weights and Measures, as to the inexpediency of changing any standard, either of length, superficies, capacity, or of weight, which already exists in a state of acknowledged accuracy; and where discrepancies are found between models equally authentic, they deem it right that such a selection should be made as will prove most accordant with generally received usage, and with such analogies as may connect the different quantities in the most simple ratios. They also concur in recommending that the subdivisions of weights and measures employed in this country be retained, as being far better adapted to common practical purposes than the decimal scale. They recommend Mr. Bird's yard of 1758 to be the legal standard of length, and approve of the means of its recovery by means of the seconds pendulum in the event of the standard being lost or injured, and that superficial measures remain as now defined by law. Also the brass weight of two pounds made by Mr. Bird in 1758 be considered authentic, and one half of this weight be the imperial standard pound Troy, consisting of 5760 grains, and that 7000 grains Troy be declared to be the pound Avoirdupois. The committee are, on the whole, induced to believe that the gallon of England was originally identical for all uses, and that the variations have arisen, in some cases, from accident, and in others from fraud. They also agree with the commissioners that the measures may be brought back to an equality, and at the same time made to bear a simple relation to the standard weight. And as it has been ascertained that the pint contains twenty ounces Avoirdupois of distilled water, the cubic inch thereof weighing 252·456 grains in air, at the mean height of the barometer, they recommend the pint to be taken as a basis, which will make the imperial gallon to contain 277·276 cubic inches of distilled water, weighing exactly ten pounds Avoirdupois. They conclude by recommending that leave be given to bring in a bill for declaring these standards of length, of capacity, and of weight, to be the imperial standards for Great Britain and Ireland, and for its colonies and dependencies; and they recommend that several copies of the standards be made with the utmost possible accuracy for the use of the Exchequer, for the three capitals, for the principal foreign possessions, for the government of France, in return for the communication of their standards, and especially for the United States of America, where the committee have reason to believe that they will be adopted, and thus tend, in no small degree, to facilitate the commercial intercourse, and by so doing to consolidate a lasting friendship between the two great nations of the world, most assimilated by their language, their laws, religion, customs, and manners."

In the course of this year, 1821, Mr. John Quincy Adams, Secretary to the United States, by order of the Senate, made and published an important report on weights and measures, from which the following brief abstracts are taken. He observes that "When Méchain and Delambre were employed by the National Assembly of France to make an admeasurement of the arc of the meridian to an extent which had never before been attempted, and to weigh distilled water with an accuracy which had never before been effected, Nature, as if grateful to those exalted spirits, who were devoting the labour of their lives to the knowledge of her laws, not only yielded to them the object which they sought, but disclosed to each of them another of her secrets.

"She had already communicated, by her own inspiration, to the mind of Newton that the earth was not a perfect sphere, but an oblate spheroid, flattened at the poles; and she had authenticated this discovery by the results of previous admeasurements of degrees of the meridian in different parts of the two hemispheres. But the proportions of this flattening, or, in other words, the difference between the circles of the meridian and equator, and between their respective diameters, had been variously conjectured from facts previously known. To ascertain it with greater accuracy was one of the tasks assigned to Delambre and Méchain, for, as it affected the definite extension of the meridian circle, the length of the mètre, or aliquot part of that circle, which was to be the standard unit of weights and measures, was also proportionably affected by it. The result of the new admeasurement was to show that the flattening was of $\frac{1}{334}$; or that the axis of the earth was to the diameter of the equator as 333 to 334. Is this proportion to the decimal number of 1000 accidental? It is confirmed as matter of fact by the existing theories of nutation and precession, as well as by experimental results of the length of the pendulum in various latitudes. Is it also an index to another combination of extension, specific gravity, and numbers hitherto undiscovered? However this may be, the fact of the proportion was, on this occasion, the only object sought. The fact was attested by the diminution of each degree of latitude in the movement from the north to the equator; but the same testimony revealed the new and unexpected fact that the diminution was not regular and gradual, but very considerably different at different stages of the progress in the same direction, from which the inference seems conclusive that the earth is no more in its breadth than in its length, perfectly spherical, and that the northern and southern hemispheres are not of dimensions precisely equal."

After this slight digression, Mr. Adams proceeds to observe that, "Considered as a whole, the established weights and measures of England are but the ruins of a system, the decays of which have been often repaired with materials adapted neither to the proportion nor to the principles of the original construction. The metrology of France is a new and complicated machine, formed upon principles of mathematical precision, the adaptation of which to the uses for which it was devised is yet problematical, and abiding, with questionable success, the test of experiment.

"To the English system belong two different units of weight, and two corresponding measures of capacity, the natural standard of which is the difference between the specific gravities of *wheat* and *wine*. To the French system there is only one unit of weight and one measure of capacity, the natural standard of which is the specific gravity of water. The French system has the advantage of unity in the weight and in the measure, but has no common test of both; its measure gives the weight of water only. The English system has the inconvenience of two weights and two measures; but each measure is, at the same time, a weight. Thus the gallon of wheat and the gallon of wine, though of different dimensions, balance each other. A gallon of wheat and of wine, each weighs eight pounds Avoirdupois."

"The litre, in the French system, is a measure for all grains and for all liquids; but its capacity gives a weight only of distilled water. As a measure of corn, of wine, or of oil, it gives the space which they occupy, but not their weight. Now as the weight of these articles is quite as important in the estimate of their quantities as the space which they fill, a system which has two standard units for measures of capacity, but of which each measure gives the same weight of the respective articles, is quite as uniform as that which, of any given article, requires two instruments to show its quantity—one to measure the space it fills, and another for its weight. In the difference between the specific gravities of corn and wine, nature has also dictated two standard measures of capacity, each of them equiponderant to the same weight. This diversity existing in nature, the Troy and Avoirdupois weights, and the corn and wine measures of the English system, are founded upon it. In England it has existed as long as any recorded existence of man upon the island; but the system did not originate there, neither was Charlemagne the author of it. The weights and measures of Greece and Rome were founded upon it. The Romans had the *mina* and the *libra*, the nummulary pound of 12 ounces, and the commercial pound of 16. The Avoirdupois pound came through the Romans from the Greeks, and through them, in all probability, from Egypt (or Tyre). Of this there is internal evidence in the weights themselves, and in the remarkable coincidence between the cubic foot and the 1000 ounces, Avoirdupois, of water, and between the ounce, Avoirdupois, and the Jewish shekel; and if the shekel of Abraham was the same as that of his descendants, the Avoirdupois ounce, &c., may, like the cubit, have originated before the flood."

"The result of these reflections is, that the uniformity of nature for ascertaining the quantities of all substances, both by gravity and by occupied space, is a uniformity of *proportion*, and not of *identity;* that instead of one weight and one measure it requires two units of each, *proportioned* to each other; and that the English system of metrology, possessing two such weights and two such measures, is better adapted to the only uniformity applicable to the subject, recognised by nature, than the new French system, which, possessing only one weight and one measure of capacity, identifies weight and measure only by the single article of distilled water; the English uniformity being relative to the *things* weighed and measured, and the French only to the *instruments* used for weight and mensuration. The habits of every individual inure him to the comparison of the definite portion of his person with the existing standard measures to which he is accustomed. There are few English men but could give a yard, a foot, or inch measure from their own arms, hands, or fingers with great accuracy.

But they could not give the mètre or decimètre, although they should know their dimensions as well as those of the yard or foot.

"The decimal principle can be applied only with many qualifications to any general system of metrology; that its natural application is only to *numbers*, and that *time, space, gravity*, and *extension*, inflexibly reject its sway. It is doubtful whether the advantage to be obtained by any attempt to apply decimal arithmetic to weights and measures would ever compensate for the increase of diversity, which is the unavoidable consequence of change. Decimal arithmetic is a contrivance of man for computing numbers, and not a property of *time, space*, or *matter*. Nature has no partialities for the number *ten*, and the attempt to shackle her freedom with it will for ever prove abortive." [1]

In the year 1824 the Act of Imperial Weights and Measures (5 Geo. IV. c. 74) was passed, founded on the preceding reports for ascertaining and establishing uniformity in weights and measures.

The preamble of the Act declares the necessity that weights and measures should be just and uniform for the security of commerce, and for the good of the community; and recognises the provision of the Great Charter, wherein it is provided that there shall be but one measure and one weight throughout the realm. Of nearly sixty Acts of Parliament of former reigns, some have been wholly or partly repealed, and others have been confirmed by this new Act for Imperial Weights and Measures. This Act brought back into use very nearly the ancient corn measure which had been superseded for between three and four centuries; the new measure being somewhat greater than the old measure by nearly one part in one hundred of the new measure.

This Act directs that after 1st May, 1825, the standard brass weight of one pound Troy weight, made in the year 1758, then in custody of the Clerk of the House of Commons, shall be the standard measure and unit of weight, and denominated the imperial standard Troy pound, from which all other weights shall be derived; and that one-twelfth part of this pound shall be an ounce, and one-twentieth part of the ounce shall be a pennyweight, and that one twenty-fourth part of a pennyweight shall be a grain, so that 5760 grains make a pound Troy: and that 7000 grains make a pound Avoirdupois, and that one-sixteenth part of this pound shall be an ounce, and that one-sixteenth part of this ounce shall be a dram.

And in case of loss or injury to the standard, since it has been ascertained by the late commissioners that a cubic inch of distilled water, weighed in air by brass weights at the temperature of 62° of Fahrenheit's thermometer, the barometer being at 30 inches, is equal to 252·458 grains, of which the imperial Troy pound contains 5760, it was ordered that a new standard Troy pound shall be made bearing the same proportion to the weight of a cubic inch of distilled water as the said standard pound Troy.

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[1] In the Minutes of Evidence taken before the Select Committee of the House of Commons, on Weights and Measures in 1862, the following question by the Chairman, and answer by James Yates, Esq., F.R.S., are printed, p. 54:—

"499. Did not the Americans, in the year 1821, establish an inquiry by Comission into the desirability of adopting a decimal system?—Yes; it was brought before them, and they appointed Mr. John Quincy Adams to inquire into the subject and make a report; and this is the report which he delivered to Congress (*handing book to the Committee*), and the main substance of his report is, that he most decidedly and in the strongest terms recommends the metric system."

If Mr. Yates had read Mr. Adams's report, he could only have made out his answer on the principle of the non-natural employment of language, as practised by a certain school of Oxford theologians.

It is also ordered that the imperial gallon shall be the only standard unit or measure of capacity, as well for liquids as for dry goods, which are measured like liquids and not heaped, and that from this measure all other measures shall be derived. That the imperial gallon shall contain ten pounds Avoirdupois weight of distilled water, weighed in air, at the temperature of 62° of Fahrenheit's thermometer, the barometer being at 30 inches. That a measure shall be made of brass, of such contents, and shall be the imperial standard gallon measure, and that all other measures shall be taken as parts or multiples, or certain proportions of the imperial gallon; that the quart shall be the fourth part of such gallon, and a pint shall be one-eighth of such gallon, and that two gallons shall be a peck, and eight gallons a bushel, and eight bushels a quarter of corn or other dry goods not measured by heaped measure. And in using these measures, the goods measured shall be stricken with a round stick or roller, straight, and of the same diameter from end to end.

And further for heaped measure the standard measure shall be the bushel containing eighty pounds Avoirdupois of water distilled, being made round with a plane and even bottom, and being 19½ inches from outside to outside. And in using such bushel, the goods shall be duly heaped up in the form of a cone, of the height of six inches, and the outside rim of the bushel shall be the extremity of the base of the cone; and that three bushels shall be a sack, and that twelve such sacks shall be a chaldron.

When heaped measure is used, it is declared that the bushel being in form cylindrical, and having a diameter not less than double the depth, shall contain 80 pounds Avoirdupois. It further orders that the goods measured shall be heaped in the form of a cone, the height of which shall be at least three-fourths of the depth of the measure, and the outside rim of the bushel shall be the circumference of the base of the cone.

The Act also provides that the straight line between the centres of the two points in the gold studs in the straight brass rod made by Mr. Bird, whereon the words and figures "standard yard, 1760," are engraved, shall be the imperial standard yard, or unit of length from which all other measures of extension whatsoever, whether lineal, superficial, or solid, shall be derived; and that all other measures of length shall be taken in parts or multiples, or certain proportions of the said standard yard; and that one-third part of this yard shall be a foot, and the twelfth part of such foot shall be an inch; also that the pole or perch shall contain 5½ such yards; the furlong, 220; and the mile, 1760. And further that all superficial measure shall be regulated by this yard, or by certain parts, multiples or proportions thereof, and that the acre of land shall contain 4840 square yards, being 160 square perches, poles, or rods. The Act provides, in case of the standard yard being lost or injured, that it shall be restored by means of the length of the seconds pendulum, as recommended by the Commissioners and the Select Committee.

In the same year 1824, a standard yard and a standard pound weight were, according to the Act, made and deposited in the House of Commons. At the burning of the Houses of Parliament in

October, 1834, both the imperial standard yard and the standard pound were so injured by the fire as to be quite useless as standards.

The Act of 1835, 5 and 6 William IV. was passed to render more effectual the preceding Acts. It repeals some of the provisions of the Act 5 Geo. IV., c. 74, which afford facilities for fraud, and abolishes heaped measure as liable to considerable variation. The new Act omits all regulation respecting the form of the measures of capacity. An uniformity of the ratio of the diameter and depth of cylindrical measures of capacity would appear to have been a desirable provision. It is obvious that of two measures in the form of a cylinder, the more shallow of the two will allow of a larger heap in the form of a cone above the rim of the measure, than the deeper measure of the same capacity. This uncertainty was remedied by providing that such articles of merchandise as before were sold *by heaped measure*, should in future be sold *by weight*. And the Act directs that the uncertain weight called a stone, shall henceforth consist of 14 standard pounds Avoirdupois, and a hundredweight of 8 such stones; and a ton of 20 such hundred weights. The Act also provides that all articles are to be sold by Avoirdupois weight, except gold, silver, platina, diamonds and other precious stones, and drugs sold by retail; and that such articles and none others may be sold by Troy weight.

In the year 1838 a Commission was appointed to consider the best method of restoring the standards of weight and measure which had been damaged in the burning of the Houses of Parliament. Their report, with an appendix, was printed in 1841, and presented to both Houses of Parliament by command of Her Majesty.

They considered that the provisions for the restoration of the yard by means of the length of the seconds pendulum cannot be carried out, as several of the elements of reduction were doubtful or erroneous; the reduction to the level of the sea was doubtful, the reduction of the weight of air was erroneous, the specific gravity of the pendulum was erroneously estimated, and the faults of the agate planes introduced some degree of doubt.

And as the exact weight of a cubic inch of distilled water is yet doubtful, the pound weight cannot with certainty be restored, as directed by the Act. They were, however, fully persuaded that, as several weights and measures exist which were most accurately compared with the former standards; by the use of these, the values of the original standards can be restored without sensible error. And further that it is always possible to restore standards by material copies more securely than by any experiments referring to natural constants, such as the length of a degree at any assigned latitude on the earth's surface, or the length of the seconds pendulum.

They advised that the standards of weight and of capacity should be defined, the former by a piece of metal, the latter by a certain weight of distilled water, without reference to the standard of length; but that the content of the standard of capacity as expressed by the units of cubical measure dependent on the standard of length, should be stated as the best determination known.

They advised the adoption of the Avoirdupois pound, and the abolition of the Troy pound. They considered the definition of the standard yard and pound in the Act 5 Geo. IV., c. 74, the best possible, and advised that the definition of the gallon be retained. After weighing most carefully the advantages and disadvantages of the metric system, they decided to recommend (though not with absolute unanimity) that no change be made in the value of the primary units of the weights and measures of this kingdom, or in the meaning of the names by which they are commonly denoted.[1] They also thought it undesirable to enforce the decimal system in all parts of the various scales of weights and measures, even were the attempt likely to be attended with fewer difficulties than would be experienced in this country. In their opinion the scale of binary subdivision is well adapted to the small retail transactions that seldom become the subject of written accounts, and which constitute a large part of the daily transactions in every country.

Another Commission of scientific men was appointed in 1843 for the restoration of the standards, and they made their final report to the Lords of the Treasury on 28th March, 1854.

On the general adoption of the report, an Act 18 & 19 Vict., c. 72, was passed for legalising and preserving the restored standards of weights and measures, which received the royal assent on 30 July, 1855. This Act repeals certain parts of the Act 5 Geo. IV., c. 74, and provides that the straight line, or distance between the centres of the two gold plugs or pins in the new bronze bar, deposited in the office of the Exchequer, shall be the genuine standard yard of that measure of length called a yard,[2] and the said line or distance between the said plugs or pins in the bronze bar at 62° of Fahrenheit's thermometer, shall be the imperial standard yard.[3]

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[1] One of the Commissioners, J. E. D. Bethune, Esq. (though he signed the report), differed in opinion from his colleagues, and expressed his views in a letter to the Chancellor of the Exchequer. One of his proposed changes was, that the stone, the hundredweight, and the ton, should be abandoned for weights of 10, 100, and 1,000 pounds, and adds immediately after, these words: "My own opinion of the character of these changes is that, if it were possible to substitute privately the weights and measures which I have proposed, the great mass of the population would never discover by the mere use of them that any change whatever had been made." The Commissioners report that they think it probable that not less than 30,000 tons of these weights are now in use, and that the expense of changing them for weights in the decimal scale would be between £100,000 and £200,000. Is the practical advantage worth the money, and the confusion that would follow?

[2] The following description of the new standard imperial yard as restored, is given in the preamble of the Act:—"The form adopted for the standard of length, and for all the copies thereof, is that of a solid square bar 38 inches long, and 1 inch square in transverse section, the bar being of bronze or gun-metal; near to each end a cylindrical hole is sunk (the distance between the centres of the two holes being 36 inches) to the depth of half an inch; at the bottom of this hole is inserted in a smaller hole a gold plug or pin, about one-tenth of an inch in diameter, and upon the surface of this pin there are cut three fine lines, at intervals of about the one-hundredth part of an inch transverse to the axis of the bar, and two lines at nearly the same interval parallel to the axis of the bar; the measure of length is given by the interval between the middle transversal line at one end and the middle transversal line at the other end, the part of each line which is employed being the point midway between the longitudinal lines; and the said points are herein referred to as the centres of the said gold plugs or pins." The standard yard and pound have been deposited in the office of the Exchequer at Westminster. Copies have also been deposited at the Royal Mint, at the Royal Observatory, Greenwich, and with the Royal Society of London.

In the 147th volume of the Transactions of the Royal Society, pp. 621—702, is contained a very interesting account of the construction of the new national standard of length, and of its principal copies, by G. B. (now Sir G. B.) Airy, Astronomer Royal.

[3] Mr. Whitworth, before the Lords' Committee, in 1855, exhibited an inch measure, with an apparatus for testing its length to the millionth part of an inch, and

The Act also provides that the new weight of platinum containing 7000 grains, and equivalent to the pound weight Avoirdupois, is the legal standard of weight, and named the imperial standard pound Avoirdupois, from which all other weights and measures having reference to weight are to be derived: and 5760 grains shall be a a pound Troy.

And if at any time hereafter the imperial standard yard or imperial standard pound Avoirdupois respectively be destroyed or injured, the Commissioners of Her Majesty's Treasury may cause the same to be restored, by reference to, or adoption of any of the authorised copies that have been deposited in the places named in the Act.

On the motion of Mr. W. Ewart, in the House of Commons, a Select Committee was appointed on 8th April, 1862, to consider the practicability of adopting a simple and uniform system of weights and measures, and they made their Report on the 15th July, 1862.

They recommend "that the best course to adopt is *cautiously but steadily*,[1] to introduce the French metric system into this country;" adopting its nomenclature, at first merely *legalising its use*, and then after a time *rendering it compulsory*; but that no compulsory measures should be resorted to until the metric weights and measures are sanctioned by the general conviction of the public. As the date of this period is so distant and indefinite, it may fairly be regarded as likely to be coincident with the date of the Greek Kalends.

A large amount of evidence was taken by the Committee, and different views and opinions were stated. The evidence[2] of Mr. (now Sir) G. B. Airy is distinguished by sound, practical common sense. While admitting the anomalies of the existing system of the English weights and measures, he maintains they do not occasion any very great

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insisted on the greater importance to all who are engaged in the mechanical arts of having a standard foot and a standard inch, than a standard yard. Engineers adopt the inch as the unit of length, and all fractional parts are expressed in decimals of the inch. They prefer the inch to the English yard and the French mètre. Like their Saxon forefathers, they have found its practical value, and approve of it in their daily work.

[1] The following remarks of the late Professor De Morgan on this point are not unworthy of attention:—"The progress of this question is one of those remarkable signs which show the tendency of our present politics. If the classes from which legislators and ministers are usually chosen intended, and were justified in intending, to train up the country into a pure democracy, in which the *Comitia* of the people should be the real executive government, and Parliament should exist only for the registration of their conclusions, they would deserve admiration *for the steady and cautious manner in which they are and have been proceeding.* Instead of having and acting upon a fixed opinion as to *what is wise and right*, or even professing to have it, both the Cabinet and the House of Commons give the country to understand that their business is only to apply their best endeavour to carry into effect such measures as are dictated by a sufficient pressure from without. It is held to be a good answer to a demand for the consideration of an important question, that there is no outcry for it in the country. Even the House of Commons cannot put the Government in motion without this outcry at its back. A few days after the House had voted a strong resolution in favour of the pound and mil scheme, urging its immediate adoption, the Chancellor of the Exchequer informed a deputation that the Government intended to—do nothing."

[2] Mr. Airy quoted the following case in his evidence:—In 1824 M. Damoiseau, under the authority of the French Board of Longitude, published some lunar tables on the centesimal division of the quadrant, which were found of great advantage; but the French Board of Longitude subsequently destroyed them. It was found that the people of France would not use the centesimal division of the quadrant. The work appeared again in 1828, but the stereotype was broken up, and the tables on the old system were substituted.

inconvenience in the use of them. He believes that the French system would not suit the habits of the people of this country; but being guided by no broad or symmetrical principle, but simply by the convenience of things and the use of them, he prefers the existing system, and declares he does not see how the decimal subdivision is to be carried through, as it is limited, and unsuited to the wants of retail trade. And that any interference of Government for compelling the use of foreign measures in the ordinary retail business of the country would be intolerable; that they could not enforce their penal laws in one instance in a thousand, and in that one it would be *insupportably oppressive.* He admits that there should be a legal power to use decimal parts of anything, because that in different trades and manufactures there are different measures used on the one hand and different weights on the other, which have been adopted by custom on account of their convenience. Such divisions, he adds, are likely to be adopted in other trades and manufactures, whenever advantages of a practical nature can be shown. And further, the reference to the different parts of any scale of weights and measures is made by different persons and for different purposes; and the various parts of any one scale are not usually combined in the practice of trades or of manufactures. He also remarks, it must be remembered, that there is no country which interferes so little as this country does with the education of the people, or with their internal affairs. The rule in this country is, that the people should act for themselves, but the rule of foreign countries is that their paternal government should act for the people. This makes all the difference between a people which is self-helping and one which is not. The object of legislation on weights and measures is to prevent fraud and misunderstanding, and in no case is the legislature of this country justified in interfering in a contract between man and man, except for the protection of the ignorant and the weak.

The leading journal of England of the date 9 July, 1863, thus supports the views of the Astronomer Royal:—"From a division in the House of Commons yesterday, it appears we are seriously threatened with a complete assimilation of all our weights and measures to the French system. Three years are given to unlearn all the tables upon which all our buying and selling, hiring and letting, are now done. Three years are supposed to be amply sufficient for undoing and obliterating the traditions of every trade, the accounts of every concern, the engagements of every contract, and the habits of every individual. But we very much doubt whether the general shopkeepers, who take possession of the corners of our small streets, or the greengrocers, will be able in three years to translate their accounts into decas, hectos, kilos, myrias, steres, and litres, mètres, millimètres, centimètres, and the hundred other terms extracted by our ingenious neighbours from Latin or Greek, as may happen to suit their purposes. Is the House of Commons, then, really prepared to see the votes, the reports, the returns of the revenue, the figures of the national debt, all run up in paper francs, and actually paid in gold Napoleons?"

In 1864 was passed an Act to render permissive the use of the French metric system of weights and measures.[1] The preamble declares that "for the promotion and extension of the internal as well as of foreign trade, and for the advancement of science, it is expedient to legalise the metric system of weights and measures." The convenience and utility of any system of weights and measures constitute the chief reason for retaining them, and any proposal to substitute a new for an old system which has been for ages deeply imbedded in the laws, the customs, and the habits of a people, appears to be at least a matter of doubtful expediency. It may be observed that the retail trading transactions of the mass of the population, and the exactness of scientific inquiry, do not require the same weights and measures. The weights and measures that would be proper for retail trade would be unfit for the exactness of scientific investigation. The past history of the progress of science owes little, if anything, to the French metric system for its advancement; and it is not obvious how in future it could be employed for this purpose, so as to secure, by its employment, any advantages which have not been already secured by the system long employed in England. The decimal system has been, and still is, largely employed both in measures and calculations where its use has been found to be convenient, and will continue to be so employed.

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[1] When Mr. Ewart's Bill went into Committee in the House of Commons, on May 4th, 1864, it fell dead, when one speaker expressed his desire to see an example

The decimal division of the British coinage, though very strongly advocated,[1] has not been authorised by the Act which legally *permits* the use of the decimal system of weights and measures. It is possible that in some trades and manufactures the use of it may suggest some improvements, without superseding the established system, which is understood and suited to the wants of the people.

The opinion of the late Sir J. W. F. Herschel, in 1864, on the standards of measure, is worthy of consideration:—"Whatever be the historical origin of our standards of weight, capacity, and length, as a matter of fact our British system refers itself with quite as much arithmetical simplicity, through the medium of the inch, to the earth's polar axis as the French does through that of the metre to the elliptic

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of the measure proposed for adoption. Mr. Ewart had forgotten that the construction of a perfect and just standard is the first step in legislation on such subjects.

[1] The merchants and bankers of the City of London in one of the most influentially-signed petitions which ever emanated from the city, presented in 1855, use language on this subject which is hardly that of petitioners. They say, that "the pound constitutes an English national fixed idea of value and position, and is associated with every existing contract, and every comparison of past revenue, expenditure, and price, and *must be retained.*" They say, also, that every other method, except that from the pound downward, is altogether impracticable. . . . We feel perfectly easy in our reliance on the common sense of the country, that it will not hear of the expulsion of the pound sterling from accounts, while the sovereign is to be retained as means of payment, after division by 24; that it will not hear of the mixed circulation of shillings and tenpences; that it will stick to its old and successful plan of reforming *that which is,* instead of substituting *that which has never been,* especially in matters connected with our oldest habits of estimation, usages of action, and associations of thought. . . . If, however, the shillings and sixpences could be replaced by an equivalent amount in tenpenny and fivepenny bits, in the course of a single night, and by the wand of a magician, the country would find itself in a poor state in the morning. Every idea of value would be upset; all notions of cheapness and dearness would require translation. A man who had made up his mind over night that he would go as far as £1 17*s.* for a purchase, must take pen and paper, if not more ready than usual, to find out how many francs he may venture upon. And this would last for years with many, for months with all. And for what? To avoid an alteration in the copper coin, which would amount to a little short of one farthing in sixpence.—*Professor De Morgan.*

quadrant of a meridian passing through Paris. It does so as regards our actual legal standards of weight and capacity with much more precision than the French system; and as regards that of length, with still greater, and indeed with all but mathematical exactness.

"If the earth's polar axis be conceived divided into 500,000,000 inches, and a foot be taken to consist of 12 such inches, then 100 of our actual legal imperial half-pints by measure, or 1000 of our actual imperial ounces by weight of distilled water at our actual standard temperature of 62° Fahrenheit, will fill a hollow cube having one such foot as its side. The amount of error in either case is only one part in 8000.

"The theoretical French mètre is the ten-millionth part of the elliptic quadrant above-mentioned; the theoretical litre is the thousandth of a cubic metre; and the theoretical gramme the millionth of a cubic metre of distilled water at 32° Fahrenheit. The actual error of the French legal or standard litre and gramme, or the deviation of these standards as they actually exist from their true theoretical value, is one part in 2730, and is consequently relatively nearly three times as great as the error in our own standards of capacity and weight, when referred to the earth's polar axis as their theoretical origin in the manner above stated. Our actual imperial measures of length deviate, it is true, by more than this amount from their theoretical values so defined—that is to say, by one part in a 1000; so that a correction of one exact thousandth part subtracted from the stated amount of any length in imperial measures suffices to reduce it to its equivalent in such units as correspond to similar aliquot parts of the polar axis. So corrected, the outstanding error is only one part in 64000. The actual legal mètre in use in France is, however, not immaculate in this respect, its amount of error being one part in 6400, which is ten times that which our British measures so corrected would exhibit. British commerce extends, however to Russia, British India, and Australia (besides North America), all of them superior in area; and the two last at least of equal importance, commercially speaking, with the totality of the *metricised* nations. The Russian *sagene* is an exact multiple of the English foot. The *hath*, the legal measure of length in British India, is 18 imperial inches. The Australian system is identical with our own, as is also that of North America. Taking into consideration this immense preponderance both in area, in population, and in commerce, we are not only justified in taking our stand against this innovation, but entitled to inquire if uniformity be insisted on; why, with an equally good theoretical basis, to say the least, the majority is called upon to give way to the minority."

The Act of 5 Geo. IV., c. 74, for the British imperial system of weights and measures, being no rash act of legislation, is founded on the experience of the past history of the English nation, and adapted for the practical convenience and benefit of the community. The French system was devised by their philosophers, who, having rejected all past experience, led by ideas, scarcely ever by facts, have sacrificed every practical consideration to the idea of reducing the divisions of all weights and measures to the denary scale of arithmetic.

ELEMENTARY ARITHMETIC,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION IV.

OF TIME.

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ON THE DIVISIONS AND MEASURES OF TIME.

"Time of itself is nothing, but from thought
Receives its rise; by labouring fancy wrought
From things considered, whilst we think on some
As present, some as past, or yet to come.
No thought can think on time, that's still confest,
But thinks on things in motion or at rest."—*Lucretius.*

The following brief notices are intended to mark the natural measures and divisions of time, as regulated by the apparent and real motions of the sun and the moon, and the artificial adjustments that have been made so as to secure their agreement with the orderly returns of the seasons, and to regulate the division of time for the purposes of ordinary life.

A measure has always a reference to some quantity which can be measured; and any quantity can be measured by any assumed measure of the same kind. Any length can be measured by any line assumed as the unit of measurement, as a yard, a mile, &c. But with respect to time, it is clear from its fleeting nature, being continuous, no standard measures can exist, like the measures of space or other quantities. There is, however, an analogy between the nature of a geometrical line and time, and an identity of language is employed of both by all nations. For instance, a point in a line and a point of time; the beginning and end of a line and the beginning and end of a period of time; the length or shortness of a line, and the length or shortness of time; also the extension of a line backwards or forwards exactly corresponds with the extension of a period of time into the past or the future. This language implies that the ideas of time are closely associated with the ideas of a line and motion; and that exact measures of the motion of bodies in lines may be assumed and employed as measures of time. It was by having recourse to the apparent motions of the sun and the moon in the heavens, that measures of time were at length ascertained.

The fact of the apparent daily motion of the sun round the heavens being observed to be constant and regular, led men to believe that the sun did really move round the earth, and this belief lasted for many ages. It had its origin in mistaking *apparent motion* for *real motion.* When a body at rest is seen by an observer from another body in motion, the former *appears to move* in a direction contrary to that in which the latter body is *actually moving.* As the daily revolution of the earth on its axis causes the sun to appear to move from east to west; also while the earth is actually moving in its orbit round the sun, in the direction from east to west, the sun appears to an observer on the earth to move among the stars from west to east. This distinction between real and apparent motions, not having been observed, was the cause of the erroneous belief.[1]

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[1] This erroneous belief has led to singular consequences. In the course of time, the bishops of Rome have decreed, that the belief of the motion of the earth on its

In the very brief notices of the records of creation contained in the first chapter of the Book of Genesis, it is stated that the Creator made two great lights, the greater light to rule the day and the less light to rule the night, and set them in the expanse to give light upon the earth; and that he also ordained them to be for signs and for seasons, and for days and for years.

From the earliest times the whole expanse of the starry heavens has appeared (as may be observed on a clear night) to move round the earth as stationary, during the alternate periods of darkness and light, in orderly succession. The sun, the moon, and "the wandering stars" also appeared, as they do now, to move round the earth within this vast expanse, in the boundary of which the fixed stars appear as shining points of different degrees of brightness.

The sun, the source of light, was observed to rise successively and regularly in the east, and to move round the heavens, giving light to the earth, and to go down in the west; and the period of darkness came on and lasted until the sun rose again on the next morning in the east. The sun also, besides its daily rising and setting, appeared to move round the whole circle of the heavens during the period of the four seasons of spring, summer, autumn, and winter.

The moon also, like the sun, appeared to move round the earth, and to pass through a regular succession of changes, alternately waxing and waning in brightness. These risings and settings of the sun, the changes of the moon, and the orderly succession of the seasons, marked the primitive divisions and periods of time.

It is highly probable that before the deluge the period of the month was reckoned from observing the number of days in which the moon passes through all its changes from new moon to new moon, or from full moon to full moon. The month was reckoned to consist of thirty days at the time of the deluge. For it appears from the

axis and round the sun was contrary to the Scriptures, and the following sentence of the Inquisition on Galileo in 1633, is founded on this infallible decree. "The proposition that the sun is in the centre of the world and immovable from its place, is absurd, philosophically false, and formally heretical, because it is expressly contrary to the Holy Scriptures." "The proposition that the earth is not the centre of the world, nor immovable, but that it moves, and also with a diurnal motion, is also absurd, philosophically false; and, theologically considered, at least erroneous in faith." The Papal decree is acknowledged in the following declaration printed on the seventh page of the third volume of an edition of Newton's Principia. It was edited by the Jesuit fathers Le Seur and Jacquier, and published at Geneva in 3 vols. 4to in 1742. "Newton, in this third book, assumes the hypothesis of the motion of the earth. The propositions of the author could not be explained otherwise than by making the same hypothesis. From this circumstance we have been compelled to personate the character of another; but we profess to obey the decrees made by the supreme pontiffs against the motion of the earth."

That the motion of the earth is "expressly contrary to the Holy Scriptures," cannot be maintained without perverting the plain sense of Scripture. Natural phenomena are described in the Scriptures *as they appear*, and no claims or pretences are therein found to explain the true system of the universe.

From the following extract of a letter of Cardinal Mai to the late Canon Townsend, it appears that the dogmas and decrees of Infallible Authority once issued, stand irrevocable for all time. "Sancta quidem Romana Ecclesia in dogmatibus semel definitis perstat, semperque perstabit: neque a Conciliorum quorumlibet œcumenicorum placitis unquam recedet;" that is, "The Holy Roman Church stands, and always will persist in standing, firmly on its dogmas once defined; and will never recede from the decisions of any Œcumenical Councils whatsoever."—*Canon Townsend's Journal of a Tour in Italy in* 1850.

account in the seventh and eighth chapters of the Book of Genesis, that the rising of the waters began on the seventeenth day of the second month, and continued until the seventeenth day of the seventh month, a period of five months, or one hundred and fifty days. The account also supplies evidence that the year was reckoned to consist of twelve months of thirty days each, or of three hundred and sixty days. For it states that the deluge began on the seventeenth day of the second month of the six hundredth year of Noah's life, and that he and his family left the ark on the twenty-seventh day of the second month of the following year, that is, after three hundred and seventy days, or a year of twelve months of thirty days each, and ten days.

This mode of reckoning the month and the year having descended from Noah and his sons, was naturally followed by their descendants. And in the early period of the world's history the year was reckoned according to the phases or appearances of the moon, and not according to any comparison of the apparent motions of the sun and the moon.

In the course of time it was ascertained that the lunar month of 30 days was greater than the true length of the time in which the moon passed through the changes, and that the true length was less than 30 but more than 29 days.

It was also observed that the year of 360 days was less than the period in which the sun appeared to move round the circle of the heavens, and that this period was more than 365 but less than 366 days. It will be seen that the month of 30 days and the year of 12 months of 30 days, could not be made to correspond exactly with the motions either of the sun or of the moon, but that these reckonings would always fall short of the true year. Neither could the months be made always to correspond with the seasons to which they were first adjusted, but in the space of about 34 years they would be found reckoned in opposite parts of the year.

If each lunar month did consist of an exact number of days, and each year of an exact number of lunar months, there would have been no difficulty in adjusting them with each other. But as neither the lunar month consists of an exact number of days, nor the solar year an exact number of lunar months, difficulties arose in the attempts made to adjust these different measures of time to one another.

In order to remedy the discrepancies and to make the reckonings of the seasons agree with the motions of the sun and the moon, different methods were adopted by different nations. Some regulated their years by the moon, and others by the sun, making from time to time such an addition of days as to bring the periods of reckoning to correspond with the lunar and solar motions whenever the lunar year ranged too widely from the solar year. Herodotus writes (ii. 3, 4), that he was informed by the priests at Heliopolis, the chief seat of Egyptian learning, that "the Egyptians were the first to discover the year, dividing the year into twelve parts, and this knowledge, they said, they obtained from the stars." And he adds his opinion, "that they reckon their years more judiciously than the Greeks; for these intercalate a whole month every other year, but the Egyptians, dividing the year into 12 months of 30 days each, add every year a space of 5 days besides, by which means the order of the seasons is made to return with uniformity." From the fact of Egypt having been one of the earliest of civilised nations, it is not improbable that this adjustment of the periods of the moon and of the sun is due to the ancient Egyptians.

It is unknown at what period the intercalation of five days was adopted. It was in use 1322 B.C., when Amunoph I. was king of Egypt, as an inscription of that age, preserved in the Museum of Turin, records the year of 365 days.

After the discovery that the *apparent motion* of the heavens and of the sun round the earth was caused by a *real motion* of the earth, it became necessary that the definitions of the day and the year should contain some reference to the *true* causes, as well as the *apparent* causes of these phenomena.

Instead of the sun daily moving round the earth at rest, it was found that the earth, being nearly globular, was actually revolving round its own axis, and that the natural day, the interval between two successive risings or settings of the sun, is, in fact, the time in which the earth makes one complete revolution on its axis; and that the periods of light and darkness are caused as one half of the surface of the earth in its rotation is turned towards the sun and receives its light, or is turned from the sun and does not receive it; and instead of the sun moving round the heavens, it was discovered that the earth was actually moving in an orbit round the sun, the central luminary of the system of worlds of which the earth is one ; and it was ascertained that the natural year, during which the sun appears from the earth to move once round the circle of the heavens, is, in fact, the period in which the earth completes one revolution in its orbit round the sun. It is this revolution of the earth round the sun that occasions the different lengths of the light and darkness, and the change of the seasons by the different positions of the earth in its orbit. The axis of the earth maintains very nearly the same inclination to the plane of the orbit in which its revolution takes place; but more exact observations have shown that the obliquity of the ecliptic is slowly diminishing.

Though the period of light and the period of darkness is different at different seasons of the year, yet the natural day, or the whole period of light and darkness, has always been of constant length. The days are longer and the nights are shorter in summer, and the nights are longer and the days are shorter in winter. And there is one day in summer when the daylight is longest and the darkness shortest; and one day in winter when the darkness is longest and the light shortest. Also, twice in the course of the year, both the periods of light and darkness of the day are equal. The two days on which these events take place are called the vernal and the autumnal equinox, one happening in the spring and the other in the autumn.

If the orbit of the earth were circular, there would be no difference between the successive days of the year. But as the earth's orbit is not a circle but an ellipse, having the sun in one of the foci, the earth's daily motion in her orbit is not uniform, and the sun in its apparent motion does not daily return to the same meridian of the earth after equal intervals, and consequently its apparent daily motion cannot be an exact measure of time.

From the time of Hipparchus, the intervals of the successive returns every day of any place on the surface of the earth to the same star in the heavens has been observed to be *constant* at all times of the year, and this period of time has been named the *sidereal day*. On this account the sidereal day has been adopted by astronomers as the principal unit of time. It may appear paradoxical that at whatever place on any day the earth may be moving in its orbit, the same place on its surface always returns to the same fixed star at the same moment of time. This fact arises from the vast distance of the earth from the fixed stars, which is so immense that even the diameter of the orbit of the earth, a distance of above 180,000,000 miles, is so small in comparison as to be inappreciable.

To an observer on the earth the sun appears to move round a great circle of the heavens in 365¼ days nearly, or through 59′ 8″ of a degree of this circle in one day, which motion corresponds to three minutes fifty-six and a half seconds of time. The average interval between the sun appearing to leave, and to return to the same meridian of the place, will be longer than the *sidereal day* by three minutes, fifty-six and a half seconds, and has been defined to be *the solar day*. And it is obvious that the *mean solar day*,[1] or the mean of all the solar days in one year, will be constant, but sometimes longer and sometimes shorter than an actual solar day, according to the velocity of the earth in its orbit during that day. A clock, therefore, which keeps mean solar time does not always indicate 12 o'clock at the moment when the sun passes the meridian. The mean of solar days, being constant, has been assumed as the unit of measure for the civil day.

The difference between true time and apparent solar time is greater or less according to the sun's apparent motion, or rather according to the actual angular velocity of the earth in any part of its orbit. The solar day being the interval between the sun's apparent departure from any given point in the heavens and its next return to that point, with so much more as is added to its diurnal motion eastwards, it is obvious the solar day is longer than the sidereal day.

The civil day is the same as the natural day, but in different ages and among different people it was considered to begin at different times of the day. Some nations reckoned the day to begin at sunrise, others at sunset; some from noon, others from midnight. Hipparchus, the Greek astronomer, who lived in the second century before the Christian era, reckoned the day to begin from midnight, and his example has been since followed by astronomers.

The origin of the custom of dividing the period of daylight into twelve equal parts is unknown. There is no mention or allusion made by Moses to any such division among the Egyptians at the time of the Exodus. After this event the time from sunset to sunrise was divided by the Hebrews into three equal parts, called watches. The morning watch is noted in Exod. xiv. 14, and the middle watch in Judges vii. 19. In later times, when the Romans held possession of Judea, the Hebrews adopted the Roman method of dividing the night into four watches, all of which are stated in the thirteenth chapter of St. Mark's gospel. The division of the day, the period from sunrise to sunset, into twelve equal parts, is noted in John xi. 9. The first hour began at sunrise, midday or noon was the sixth hour (John xix. 14), and the twelfth hour ended at sunset. The third hour divided the interval between sunrise and noon (Mark xv. 25), and the ninth hour between noon and sunset (Matt. xxvii. 45). The two intermediate points of time which equally divided the periods of light and darkness were noon and midnight.

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[1] The mean solar day has been described as the mean of the solar days of a year. An imaginary sun is supposed or conceived to move uniformly in the equator with the real sun's mean motion, and the interval between the departure of any meridian from the mean sun and its returning to it, is the duration of the mean solar day. Clocks and chronometers are adjusted to mean solar time; so that a complete revolution through 24 hours of the hour hand of the Astronomical clock should be performed in exactly the same interval as the revolution of the earth on its axis with respect to the mean sun.

The month having been defined to be the time in which the moon was observed to pass through its changes, is in fact the period during which the moon completes one revolution in its own orbit round the earth, and is at the same time carried with the earth in its orbit round the sun. As the moon receives its light from the sun in the same manner as the earth, the appearance of the new moon will take place when the dark side of the moon is turned towards the earth, or, more correctly, when the moon and the sun are *in conjunction*, or on the same side of the earth. And the full moon will happen when the moon and the sun are *in opposition*, or on different sides of the earth. On the second day after the conjunction the new moon is first seen as a thin curved line of light, as part of a circular arc; but the actual fact of conjunction can only be seen when the conjunction is so close as to occasion an eclipse of the sun. In early times the interval between two successive new moons was observed to be about thirty days, and this period was reckoned the length of the month, until more accurate observations showed that the exact time between two new moons was less than thirty but more than twenty-nine days. At length in modern times it was found that the average interval between two successive new moons is 29 days, 12 hours, 44 minutes, and 3 seconds; and the average interval from new moon to full moon is 14 days, 18 hours, 22 minutes, and 1½ second; and the full moon will, therefore, generally happen on the 14th or 15th day of the moon's age.

The natural day, the lunar month, and the solar year are fixed, but incommensurable with each other; thus, if the year be measured by days, it contains more than 365, but less than 366 days; if by lunar months, it contains more than 12 but less than 13; and if the lunar month be measured by days it contains more than 29 but less than 30 days.

In addition to the three natural divisions of time, the year, the month, and the day, the most ancient of the artificial divisions of time is *the week*, a period of seven days.[1] The origin of this division of time is to be referred to that period of the existence of the planet, the earth, when the first man and woman were created, as recorded in the very brief notices contained in the first two chapters of the Book of Genesis. At the beginning it was ordained by the Creator that every six successive days of human life should be assigned to ordinary work, and the seventh day be a day of rest from labour for the whole human family.

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[1] The period of seven days, the seventh day, and the number seven, are found in the traditions and languages of almost all nations, fragments of the original institution of the seventh day. It is not improbable that the name of the practice of an observance might remain among a people long after the original tradition had been lost, and any superstition might become connected with the name. The ancient Egyptians celebrated the festival of their god Apis for seven days. The time of six times seven days was the period of mortification imposed on the Egyptian priests. And their custom of mourning for the dead was extended to ten times seven days. It is recorded (Numb. xxiii. 1, 2) that Balak, the king of Moab, offered, by the direction of Balaam, seven oxen and seven rams upon seven altars. And in a later age, the Syrian general Naaman was directed by the Hebrew prophet Elisha to wash seven times in the river Jordan for the cure of his leprosy (2 Kings v. 10).

In a still later age, the most ancient writers of the Greeks recognised the sanctity of the seventh day of the month. Thus Hesiod calls "the seventh day the illustrious light of the sun." And Homer writes:—"Then came the seventh day, which is a sacred day." And Varro, in his book inscribed *Hebdomades*, observes that he had then entered upon the twelfth week of his years. It would be easy to multiply instances of the use of the number seven among peoples separated from the Hebrew nation. The tradition can only be referred to the primeval institution of the seventh day.

In the first ages of the world the seventh day of rest was observed. It is also evident from the eighth chapter of the Book of Genesis that time was reckoned by periods of seven days in the age of Noah; and there are indications in the same book that this division of time was observed during the lives of the Hebrew patriarchs.

This tradition was probably neglected or forgotten during the period of the slavery of the Hebrews in Egypt. The primitive law of a seventh day of rest was, however, recognised in the opening of the fourth precept of the Decalogue, in the words, "Remember that thou keep holy the sabbath day."

On the seventh day of rest being re-proclaimed to the Hebrews, some additions as to its observance were made for their use, commencing with the year of their settlement in Palestine. The number seven is connected and, as it were, interwoven with almost all their ceremonial laws and religious rites. Besides every seventh day as a day of rest, every seventh year was to be a year of rest to the land, and seven times seven years brought the great year of rest and release. It would also appear that the custom in the more ancient times of celebrating a marriage feast for seven days (Gen. xxix. 28) was continued to be observed by the Hebrews in later times (Judges xiv. 12, 17). The seventh day is observed by their descendants in all countries at the present time. The Hebrews, who were the first disciples of the Messiah, observed the first day of the week on which He rose from the dead, instead of the seventh day, as a day of rest, and, in accordance with this early practice, all Christian communities in succeeding times have followed their custom.

In the prophetical writings of the Hebrews, a day is used sometimes for a year, and a week of days for seven years, also (Dan. ix. 24) seventy weeks is the expression employed to mean seventy times seven years. The seventh day of rest and the time of the beginning of the natural day are both noted in the early part of the Book of Genesis. The words used to describe the two parts of the natural day, "the evening and the morning," are exactly preserved in the Greek word *νυχθήμερον* (Gen. i. 5; Dan. viii. 14; 2 Cor. xi. 25).

The ancient Hebrews reckoned the day to begin in the evening at sunset, and the length of the day was reckoned as the period between two successive settings of the sun (Exod. xii. 18; Luke iv. 40). The Hebrews appear not to have assigned any names to the seven days of the week, but simply numbered them in order.

In the Roman Fasti, Ovid makes no mention of weeks or a seventh day. Dio Cassius, who lived in the time of the Emperor Alexander Severus (A.D. 222–235), writes that the custom of dividing time into weeks was derived from the practice of the Egyptians a little before his time, and that the seven days of the week were named from the planets: Dies Solis, Lunæ, Martis, Mercurii, Jovis, Veneris, Saturni. It may be added that the first day of the week is named *ἡλίου ἡμέρα* by Justin Martyr in his "First Apology," about the middle of the second century.

The names given to the seven days of the week by our Anglo-Saxon ancestors before their conversion to Christianity continue in use at the present time. It is a singular fact that these names have descended unaltered and unchanged by the revolution which took place in England when it was subjugated by William Duke of Normandy. The names of the seven days of the week are formed by annexing the word *daeg, a day,* to the names of the Saxon deities: Sun, Moon, Tuisco, Wodin, Thor, Friga, Seater. The word *se'nnight* (seven night) is used to denote a period of seven days, and bears on its face a faint trace both of the day beginning at even, and the original appointment of the seventh day; so also its kindred *fortnight* (fourteen night), fourteen, or twice seven nights.

In addition to the divisions and measures of time already noticed, other divisions of the day have been made both for the purposes of science and the convenience of human life. It is recorded by Herodotus (ii. 109) "that the sundial,[1] the gnomon, and the division of the

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[1] The sundial is a contrivance to mark the hours of the day while the sun shines above the horizon. It is not known when the Hebrews adopted the division of the day into hours: there is no trace of it in the Pentateuch. The mention of the dial of Ahaz in the time of Hezekiah, B.C. about 730 (2 Kings xx. 9—11; Isa. xxxviii. 8), implies some sort of division, but the records give no account of the magnitude of the degrees, or what portion of time was marked by them. The word translated "hour," in Dan. iii. 19, cannot mean an hour in the sense of a twelfth part of a day, but would be more correctly rendered "moment," as the context suggests.

Anaximander, a companion and disciple of Thales, is reported by later Greek writers to have invented the sundial. He died about B.C. 547, in the sixty-fourth year of his age (Diogenes Laertius). If he derived his knowledge of the sundial from the Babylonians, he may have been the first to introduce the knowledge of it to the Greeks. The claim of the Greeks in later times to the invention of the sundial can scarcely be correct; it is, however, highly probable that the Greeks made improvements of the dials they received, just as they advanced the knowledge of geometry and other sciences which their predecessors originally brought from Egypt.

There is in the British Museum a stone block on which are marked lines which clearly indicate that they are the lines of an ancient sundial. It is supposed to be the oldest specimen of sundials known to be extant. The reader will find some interesting information in "An Inquiry into the Geometrical Character of the Hour Lines upon the Antique Sundials," by T. S. Davies, Esq., F.R.S., printed in the Transactions of the Royal Society of Edinburgh, February, 1831.

The Clepsydra, or Water-clock, was a cylindrical vessel with a small hole at the bottom. On the side of the vessel from the top downwards the hours were marked. The vessel was filled with water, which, in the space of a day, flowed out through the aperture; an index, while floating, pointed to the hours as the surface of the water gradually subsided. They were of various kinds, but all of them were inaccurate measurers of time, as the water would flow through the orifice in the bottom of the vessel more rapidly when full than when nearly empty, on account of the pressure being greater according to the depth of the water.

Ctesibius of Alexandria, in the time of Euergetes II., invented a more perfect hydraulic clock, and more simple than those which were then known. It is described by Aristotle (Prob. Sec. 16, p. 933) and mentioned by Aristophanes as well known in his time. Vitruvius affirms that Pliny relates that it was introduced into Rome by Scipio Nasica (Pliny vii. 37; ii. 76, in Horologium).

The clock as an instrument for the measurement of time was unknown to the Greeks and Romans. The invention of this complicated machine belongs to a more improved state of scientific knowledge. The invention of the clock (probably the balance clock) is ascribed to the Arabians, about the year 801 A.D.

Dante, who was born in 1265 and died 1321, mentions a clock in Italy that struck the hours, which is the earliest instance on record. A clock about 1288 was fixed in the famous clock-house in Westminster Hall. In 1292 a similar clock was constructed for the cathedral of Canterbury; Peter Lightfoot constructed a clock at Glastonbury in 1325 (Brady's "Clavis Calend.," vol. i., p. 8). In Rymer's Fœdera day into twelve parts, the Greeks learned from the Babylonians." The division of the natural day into twenty-four equal parts is of ancient date, and each of these equal parts is named an hour. The word ὥρα was not known in Greece so early as the age of Anaximander, yet it is probable that the division of the period of daylight into twelve equal parts was in use. Nor was the word *hora* known at Rome for 300 years after the foundation of the city.

The hour was also divided into sixty equal parts, called minutes, the minute into sixty equal parts, called second minutes, afterwards abbreviated into seconds, and so on. This division of the hour, &c., most probably was taken from the sexagesimal division of the degree, or the 360th part of the circumference of a circle.

In the history of mankind, certain points of time, called epochs, have been noted, from which periods of time have been reckoned, as the date of the Exodus from Egypt by the Hebrews, of the First Olympiad by the Greeks, of the Building of the City of Rome by the Romans, or of the Birth of Christ. A series of years not defined but reckoned from some fixed epoch is called an era, as the Christian era, and this differs from a *period* of time, which is supposed to have its beginning and ending fixed.

It appears the more desirable plan to adopt the chronology of the Hebrew Scriptures in connecting the history of ancient nations, as far as the correctness of it can be determined. The early periods of Grecian and Roman history are involved in fabulous uncertainty. It was the object of heathen priests to invest in mystery and in obscurity the events of early times, the origin of their rites and superstitions, and the exploits of those heroes they worshipped as gods. They were careful also to refer events to an antiquity beyond all record, and by artifice and ingenuity to keep the minds of their deluded votaries from any scrutiny into the truth and origin of the superstitions they found advantageous both for their own power and profit.[1]

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mention is made of a protection granted by Edw. III. in 1368 to two Dutchmen who were clockmakers to exercise their trade in England. And it is not improbable that the clock set up at Westminster in 1368 A.D. was constructed by these artists. It was, however, found to be imperfect. Richard Walingford of St. Albans, in the reign of Richard II., 1377—1399, made a clock for the Abbey at that place. The clock set up at Hampton Court Palace, 1510, was the first which kept time with regularity. Clocks were made portable about 1530, but were not in general use till 1631, and the balance clock, then in use, received its greatest improvement in the substitution of the pendulum for the balance by Huyghens, in 1649. The idea of employing the pendulum was first suggested by Galileo, who had noted the regularity and equality of its vibrations. Watches were invented at Nuremburg in 1477 by a German, and appear to have undergone considerable improvements before 1530. They were brought to England in 1577, and about eighty years after that time Hooke invented the spring pocket watches, which were followed in 1676 by Barlow's repeaters. —See Treatise on Watch-work, by the Rev. H. L. Nelthropp, 1873.

[1] From the fact of the early prevalence of the worship of the host of heaven, the aspects and motions of the sun, moon, and the planets engaged the attention of mankind. In the progress of idolatry, powers and influences were ascribed to the planets and constellations, supposed by their aspects to forebode good and evil to men. Those who studied the stars imposed on the ignorance of the people and framed the science of Astrology. This superstition prevailed in the neighbouring countries of Judea (Jer. x. 2). It was common in Chaldea (Dan. ii. 2). It prevailed long before in Egypt, and was professed both in Greece and Rome; and the superstition is still found in almost every country. That people under the influence of such mistaken notions should be alarmed at eclipses, as Nicias was (Thucyd.

The dates recorded of some important events in the most ancient histories are uncertain. In different writings the same events are referred to different dates, and even in different copies or versions of the same history, errors in numbers have crept into the records, whether from design or the inaccuracy of copyists, it is impossible to affirm. According to the genealogies in the manuscript copies of the Hebrew Pentateuch, the deluge happened 1656 A.M. The Samaritan text places this event 1307 A.M.; but the Greek Septuagint makes it 2262 A.M., and Josephus 2256 A.M. The date adopted in England is that of the Hebrew Pentateuch. Dr. Lloyd, Bishop of St. Asaph, fixed the epoch of the creation at 4004 B.C., differing by four years from that of Archbishop Usher, who fixed it at 4000 B.C.

The earliest mode of reckoning periods of time was by generations, or the times which intervened between the birth of a father and his first son. The periods from the creation to the deluge, and from the deluge to the birth of the patriarch Abraham, the head of the Hebrew race, are defined in this manner in the fifth and eleventh chapters of the Book of Genesis.

The periods of the reigns of kings were in course of time employed, where communities had adopted the kingly form of government. The periods of the reigns of the Hebrew kings, and after the disruption of the kingdom, of the kings of Israel and Judah, were adjusted in this way, until the captivity of the former kingdom by Shalmaneser, 721 B.C., and of the latter by the king of Babylon, 588 B.C.

The time of the appearance of the new moon which happens near the vernal equinox became at a very early period a remarkable epoch, and by divine command was reckoned the first day of the year of the Hebrews. It was the day of their deliverance from bondage in Egypt, and was ordained to be observed in successive generations as a memorial of that event. Their departure from Egypt began on the night of the full moon, on the 14th day of that month. The fourteenth day of the moon "at even" was the evening of the full moon, or the evening preceding it. This first month of their year always began with that new moon which came nearest to the vernal equinox, and this new moon would fall sometimes before, and sometimes after, the equinox.

This epoch regulated all their festivals and religious observances, while they retained the epoch of the beginning of the year at the autumnal equinox for the regulation of civil affairs. The Hindus reckon by lunar years the times for religious purposes, and by solar years for civil purposes.

vii. 50), need occasion no surprise. These phenomena being regarded as portending the overthrow of armies and the revolution of states, were carefully observed. But notions of planetary influence were not confined to ancient times. The rebellion in 1745 in Scotland was supposed by many to have been foretold by a comet that appeared the preceding year. The hieroglyphics of some almanacs, regarded by many as infallible adumbrations of the future, are striking proofs that the science of astrology, whence the materials for prognostications are professedly obtained, is by no means consigned to oblivion. In England, Queen Elizabeth seems to have been so fully satisfied of the truth of astrology, that she caused the Parliament to enact a law (23 Eliz., c. 2), ordaining the penalty of death to any person found guilty of casting her Majesty's nativity. The science, so-called, of judicial astrology (or rather superstition) has had many admirers since her days, and still survives, though chiefly in the persons of crafty and ignorant pretenders, who, dealing in imposture and casting nativities, impose upon the ignorance and credulity of the simple.

In the early ages of the Grecian States, the traditions which the settlers brought with them appear to have been of the most primitive character. Their years were divided into seed time and harvest, and their days into the times of labour and rest. The time from sunrise to sunset was divided into three parts, the morning, midday, and evening. In the age of Homer, lunar months were in use among them (Odys. xiv. 162), and the Athenian year was reckoned by the periods of the moon, and it was long after this time before any attempts were made to adjust the motions of the moon and the sun. The traditional accounts extant of the sages of Greece in their travels in search of wisdom, about 600 B.C., clearly show that scientific knowledge had not then begun to be cultivated in Greece. Thales of Miletus, and Solon, his contemporary, both visited Egypt as the chief school of knowledge in their time. Herodotus has recorded (i. 74) that Thales foretold to the Ionians an eclipse of the sun, which has been determined to have taken place in the year 585 B.C. Thales taught that a revolution of the moon did not exceed thirty days, and assigned the year to consist of twelve months of thirty days each. This he probably learned in Egypt. Herodotus (ii. 3, 4) writes that he himself acquired much information from the priests at Memphis, and that he went to Thebes, and to Heliopolis, the great seat of learning, expressly to try whether the priests of these places agreed in their accounts with the priests of Memphis.

Solon found the adjustment of the year by Thales was erroneous, and settled the twelve months of the year to consist of twenty-nine and of thirty days alternately. This method made the year to consist of 354 days. Both Thales and Solon devised different modes of adjusting the motions of the sun and the moon, but neither of them appears to have been successful.[1]

Meton was an astronomer of Athens at the time of the Peloponnesian war, and refused to sail with the expedition to Sicily, foreseeing the calamities that were likely to follow that rash enterprise. He had observed that the motion of the moon fell short of the motion of the sun by some hours every year, and this difference, scarcely perceivable in a small number of years, he foresaw would, in the course of a long period of years, entirely invert the reckoning of the seasons, and it would be found that the season of summer would really happen in winter, and the season of spring in the autumn. He succeeded in his attempts to adjust the motions of the sun and of the moon more accurately than before, and maintained that the lunar year and the solar year could begin at the same time, or that the sun and the moon could begin to move from the same point in the heavens. He calculated that when the sun had finished nineteen periods of its yearly motion, and the moon 235 of its monthly periods, both the sun and the moon would return nearly to the same position in the heavens in which they had been nineteen years before; or, in other words, after the completion of this period, the conjunctions, oppositions, and all other aspects of the moon, would fall on the same days of each succeeding year as of the preceding nineteen years. As the solar year did not contain an exact number of lunar months, he found out, 432 B.C., that by intercalating seven lunar months in nineteen lunar years there would be 235 lunar months in nineteen solar years very nearly, so that the times of celebrating the Grecian games and festivals could be adjusted both to the new and full moons, and to the equinoxes and solstices. It was afterwards discovered that, the moon and the sun beginning to move from the same part of the heavens, would not at the end of nineteen years be exactly together, but that the moon would be in advance of the sun, and in 310 years the new moon would appear at the end of this period one day in advance of the time reckoned at the beginning of the succeeding period.

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[1] The year of the ancient Greeks was luni-solar, consisting of 12 months of 30 days, each day half daylight and half darkness, as is clear from several testimonies, and is represented by the riddle of Cleobulus :—

Εἷς ὁ πατήρ, παῖδες δὲ δυώδεκα· τῶνδὲ ἑκάστῳ
Παῖδες τριήκοντα, διάνδιχα εἶδος ἔχουσαι·
Αἱ μὲν λευκαὶ ἔασιν ἰδεῖν, αἱδ' αὖτε μέλαιναι·
'Αθάνατοι δέ τε οὖσαι, ἀποφθινύθουσιν ἅπασαι.

In order to remedy this discrepancy, Calippus, 330 B.C., proposed four times nineteen years instead of nineteen years; but this amended scheme did not completely bring the motions of the sun and the moon into conformity. A small difference having been observed at the end of this time, Hipparchus, 146 B.C., proposed a period of eight times nineteen, or 152 years. And it may be remarked, that these three astronomers in their successive attempts, approached nearer to success in adjusting the motions of the sun and the moon than any of their predecessors.

The Olympic games were celebrated by the Hellenic people every fourth year at Olympia, a city of Elis, on the banks of the Alpheus. The time of the first institution of these games lies somewhere in the mythical period of history. These games, after some interruption, were restored by Iphitus about 884 B.C., and celebrated every fourth year at the time of full moon at the summer solstice (Pind. Ode iii.) The interval of four years between two celebrations was called an Olympiad. It was not, however, until long after this date, that Olympiads were employed for arranging the dates of historical events. When the Olympiads were adopted as a measure of time, the date of the first Olympiad was reckoned from that year in which Corœbus was victor, and its date is placed 108 years after the revival of the games by Iphitus, which year corresponds to 776 B.C. Strabo writes that the early historians of Greece were ill-informed and credulous; and it is difficult to distinguish what is mythical from what is historical. It was not until later times that any exact mode of fixing the order of events by dates came into use.

The method of reckoning by generations and the succession of kings was variable, and likely to lead into error, if the early Greek historians wrote from tradition without any original records and authentic memorials to guide them. In the poems of Homer and Hesiod no allusions are made to Olympiads. In the histories of Herodotus and Thucydides, the dates of important events are not fixed by reference to the Olympiads or to any other epoch; and even after the use of these had been established, ancient writers appear not to have been much guided by them, and consequently there is room for doubt in their statements of the exact times when events happened.

Dionysius of Halicarnassus observes that Herodotus in his histories has followed the order of events, but Thucydides, in his history of the Peloponnesian war, the order of time; that while Thucydides has divided his single subject into many parts, Herodotus, having embraced many subjects, has formed them into a single whole.

Thucydides in recording events keeps to the order of the years of the war, and fixes the date of its beginning (book ii. 2) by naming several events, one of which is the Archonship of Pythodorus at Athens. With respect to the events which happened long before the war (book i. 13, 19), he reckons back from the end of the war.

Timæus, who lived in the time of Ptolemy Philadelphus, 283 to 245 B.C., appears to have been the first who applied Olympiads to fix the dates of events in history. Even so late as 283 B.C., the Arundelian marbles make no mention of Olympiads. Eratosthenes, 191 B.C., and Apollodorus, 115 B.C., arranged events by Olympiads and the order of succession of the kings of Sparta.

The custom of reckoning from the first Olympiad appears to have been continued to 312 A.D.[1] In memory of the victory gained by Constantine over Maxentius, on the eighth of the Calends of October, 312 A.D., by which entire freedom was given to Christianity, the Council of Nice ordained that the accounts of years should no longer be kept by Olympiads, but that instead thereof, the Indiction should be used by which to reckon and date their years. And after this date the public mode of reckoning throughout the Roman Empire, both in the east and in the west, was by periods of 15 years, called Indictions.[2]

The chronology of early Roman history has been shown to be still more uncertain than that of Greece. In fact, the early history of most nations is obscure and doubtful, and hence the confusion and the contradictions which are met with in the accounts which have come down to modern times. The epoch, A.U.C., of the foundation of the City of Rome comes next in order to that of the Olympiads.

Terentius Varro makes this epoch to coincide with the 23rd year of the Olympiads, or 753 B.C., which has been generally accepted as the true date of that event. The older records of Rome were burnt by the Gauls in their attack on the city about 388 B.C. In early times of the Republic, the Romans distinguished their years by the names of the consuls for each year in succession. The period of five years was named a lustrum, and marked the intervals of taking the census of the people, first by the kings and afterwards by the consuls, but after the date 310 A.U.C., it was taken by the censors.

The ancient Roman year consisted of 10 months, and was reckoned to begin with that month which is now the third month of the English calendar. Numa Pompilius is reported to have reformed the calendar by adding two months to the year, and thus making it to consist of 12 months or 354 days, according to the course of the moon. These two months were named Januarius and Februarius. He afterwards added a day to the month of January, and so made the year to consist of 355 days. But as 10 days were wanting to make the lunar year correspond with the solar year of 365 days, he ordered in every third year a month (called *mensis intercalaris*) to be inserted in the calendar according to the discretion of the Pontifices, and by this authority they made the years longer or shorter as was found from circumstances most convenient. If Numa had retained the lunar year of 354 days, and not added one day to the month of January, his method of intercalation would have made the year as regular as that of Julius Cæsar. But by adding one day to the month of January, each year deviated from the solar year one whole day too much; and this irregularity he might have easily corrected by omitting eight days of the intercalary month every eighth year. This, however, was not done, and the progression of all the months of the year relative to the seasons continued to go onward.

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[1] Any year of a given Olympiad may be converted into the corresponding year B.C. or A.D. by multiplying the complete Olympiad by 4, and adding 1, 2, or 3 to the product, according as the given year may be the first, second, or third of the Olympiad. If this number be *greater than* 776, then the excess of this number above 776 gives the year A.D. If, however, this number be less than 776, the excess of 776 above this number increased by unity will give the date B.C.

[2] The Roman Indiction was instituted by Constantine, surnamed the Great, and was properly applied to certain tributes which were ordered by imperial edicts to be paid every fifteen years. These tributes, or Indictions, were sometimes found to be burdens too oppressive to be endured. The name Indiction was afterwards used simply to denote a period of fifteen years, and was adopted by the church.

The Decemviri about the year 452 B.C. ordered the month of February to be reckoned next after the month of January, and it has held this position from that date down to the present time.

The method of arbitrary intercalation in course of time was calculated to produce confusion and disorder in the reckoning, so that the months became removed from their proper seasons; the winter months being carried back into the autumn, and the autumnal months into the summer. This arrangement of the year, notwithstanding its imperfections, was continued until the times of Julius Cæsar, who resolved to remedy the confusion by abolishing the cause of it, namely, the arbitrary use of intercalation by the Pontifices. For this purpose, with the aid of Sosigenes, an Egyptian astronomer in the year 707 A.U.C., or 47 B.C., he adjusted the year according to the course of the sun, and assigned to each month the days they still retain. He found that the reckoning of the months had receded from their proper seasons, December coming on in September, and September in June. In order to bring the seasons forward he found it necessary to devise a year of fifteen months or 445 days (called the last year of confusion), so as to make the ensuing year, 708 A.U.C., begin on the first day of January, and so proceed regularly afterwards.

As the solar year consists of 365¼ days, Julius Cæsar ordered that the months should be reckoned, and the civil year regulated from the course of the sun and not of the moon, and disposed of the 5 days among the months, making them to consist, some of 30 days and some of 31 days, except the month of February, which, in common years, should still retain its number of 28 days. And to account for the quarter of a day over 365 days, he calculated that the intercalation of one day every four years in the month of February would bring the new scheme into concert with the order of the seasons. This intercalary day followed the 6th of the Kalends of March in the Julian Calendar, and was called bis-sextilis, the sixth day of the Kalends of March twice reckoned. From this fact every fourth year has been called bissextile, or leap year with us, one day having been passed or leaped over without reckoning in the Julian Calendar. It would not, however, have been strictly correct to say, according to the Roman mode of counting, that leap years had one day more than common years. In leap year, when the month of February consists of 29 days, both the 24th and 25th days of that month were marked in the Julian Calendar as the sixth day before the Kalends of March, and these two days were reckoned as one day, or one real day being leaped over, and the year was called bis-sextilis, as having the sixth day twice in one month.

Soon after the reformation of the calendar Julius Cæsar was killed, and his scheme was so imperfectly understood by the Pontifices, that instead of making the fourth, they made the third year bissextile. This error was discovered 37 years after the epoch of the Julian correction, when 13 intercalations had taken place instead of 10, and the year began 3 days too late. The error was corrected by the Emperor Augustus ordaining that the following 12 years should be of 365 days each, dropping the 3 intercalary years, so that there should be no leap year till 760 A.U.C., or 7 A.D. From that time the account has been kept free from error, and the Roman year has been adopted by almost all Christian nations, with no other variation than taking the epoch from the birth of Christ instead of from that of the foundation of the city of Rome.

The name of the month *Sextilis* was changed to *Augustus* (our *August*) in honour of the Emperor Augustus, and this name it has ever since retained. The name of the month *Quintilis* was also changed to *Julius* in honour of Julius Cæsar, who had been born in that month. The names of the Roman months are retained in the English Calendar. The subdivision of their months was different, as they counted the order of their days backwards instead of forwards, making a threefold division of the days of each month into Kalends, Nones, and Ides; our subdivision is into weeks, and reckoned in order forwards.

The Roman Calendar, thus reformed by Julius Cæsar, by founding his corrections on the length of the solar year (called also the Julian year), was ordered by the edict of the Dictator himself to be adopted and used throughout the empire; and after its amendment by Augustus continued to be used in Europe without any variation down to the year A.D. 1582.

From the time of the Exodus the fourteenth day of the moon has been reckoned by the Hebrews the full moon. The feast of the Passover was ordered by Moses to be celebrated at the time of full moon, and the Hebrews who became Christians would naturally celebrate the festival of Easter, the day of the resurrection of the Messiah, on the same day as the Hebrew Passover (1 Cor. v. 7; xv. 20), and there can be no doubt that the Christians of the East celebrated Easter on the fourteenth day of the moon. It appears that the Christians of the West, about the middle of the second century of the Christian era, celebrated Easter on the first Sunday after the fourteenth day of the moon. Both the Eastern and the Western Christians declared they were guided in their practice by apostolic tradition, and a schism arose between them in consequence of this difference of time in observing the festival.

The question which gave rise to this controversy was deemed of sufficient importance to be brought under the consideration of the Council of Nice, 325 A.D. After their deliberations, the Council sent an epistle (still extant) to the Church of Alexandria, declaring their judgment in the following form:—"We send you the good news concerning the unanimous consent of all in reference to the celebration of the most solemn feast of Easter, so that all the brethren in the East, who formerly celebrated the festival at the same time as the Jews, will in future conform to the Romans and to us, and to all who have of old observed our manner of celebrating Easter."

The fathers at the Council give no account of the reasons by which they ordered the Western rather than the Eastern manner to be preferred, nor even whether they consulted the astronomers of Alexandria.

As the Julian year exceeds the true solar year by about 11′ 15″, the excess in four years amounts to 45′. This excess of 45′ in the period of 131 years amounts to one day and 35′; and hence originated the difference of *the* day, when Sosigenes observed the equinox, from that in which it was observed in 325 A.D., for which the Nicene Council could not account.

The cause of that difference was not then ascertained. In fact, at that time astronomical instruments had not been invented by means of which small differences of times and motions could be measured and estimated. It was only after long periods, when the error had accumulated to several days, that it became so apparent as to require correction, as at the time of the Julian correction of the calendar, and at the settlement of the rule for finding Easter at the Council of Nicæa. The cause of this difference was not ascertained at the Council, though the fact was apparent, of the vernal equinox having gone back four days.

It is reported that bishops from the British Isles attended this Council, but it appears that the British Christians afterwards adhered to the Eastern mode of observing the festival of Easter. For when Augustine came on his mission to the Anglo-Saxons in 596 A.D., or more than 200 years after this Council, he demanded that the British Christians should conform to the Roman practice of observing Easter. They firmly declined to submit, clearly proving that the Christian Church in Britain existed before the mission of Augustine.

The epoch of the Christian era was not used in the computation of time for several centuries after the time of the birth of the Messiah. Dionysius, a native of Scythia, called Exiguus, on account of his stature, was the founder of this epoch. He was a monk, a man of great learning, and he became an abbot at Rome in the early part of the sixth century. He calculated that the time of the birth of the Messiah coincided with the 25th December, 753 A.U.C., which is now considered to be four years too late. In two letters of Dionysius, one written in 526 A.D., and the other the year after, is described his reformation of the calendar. The alteration proposed he thus states:—"But since Cyrillus began his first cycle from the 153rd year of Diocletian, and finished the last in the 247th year; we, beginning from the 248th year of that tyrant, rather than prince, refuse to connect the memory of a blasphemer and persecutor with our cycles, but rather choose to note the dates of our years from the incarnation of our Lord Jesus Christ."[1]

Instead of reckoning the beginning of the Christian era from the 25th December, A.U.C. 753, he made the 1st day of January, the first month of the Julian year, to be the beginning of the Christian era, so that the year 1 A.D. (Anno Domini) was made to coincide with 754

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[1] "Quia vero S. Cyrillus primum Cyclum ab anno Dioclesiani centesimo quinquagesimo tertio cœpit et ultimum in ducentesimo quadragesimo septimo terminavit, nos a ducentesimo quadragesimo octavo anno ejusdem tyranni potius, quam principis, inchoantes, nolumus circulis nostris memoriam impii et persecutoris innectere, sed magis elegimus ab incarnatione Domini nostri Jesu Christi annorum tempora prænotare."—*Petavii Doctr. Temp. sub finem.*

A.U.C. It will be obvious this ancient mode was different from the modern mode of reckoning. The year before, after the Roman manner of reckoning, being considered the year of the birth, the preceding year was reckoned the second year before the event. It will be obvious, as the date A.D. did not begin to be reckoned from 0, but from 1, in calculating any number of days between any year before and any year after the epoch of the Christian era, one year must be subtracted from the sum of the dates.

The pontificate of Gregory XIII. was distinguished by the reformation of the calendar, but was indelibly disgraced by his approbation of the massacre of the Protestants in France on St. Bartholomew's Day in the year 1572. His infamy is perpetuated by the medal he had struck to commemorate that deed of bloodshed.[1]

The eleven minutes' excess of each Julian year above the tropical year had so accumulated between the years 325 and 1582, A.D., that the reckoning of the vernal equinox took place 10 days too early. At the time of the Council of Nicæa in 325, the vernal equinox fell on 21st March, and the Council decreed that the festival of Easter should be celebrated on the first Sunday after the full moon that happened after this equinox. In 1582 the vernal equinox, according to the reckoning, fell on 11th March; and the festival of Easter, coming 10 days too early, occasioned also irregularity in the times of celebrating other festivals of the Church.

In order to bring back the reckoning to the true time of the equinox, Pope Gregory XIII. caused ten whole days to be omitted or left out of the month of October, in the year 1582, so as to make the years in future to agree with the seasons, and the vernal equinox to be reckoned at the right time. And in order to prevent the recurrence of this error at any future time, it was fixed that, besides the Julian correction of every fourth year being a bissextile, the last year A.D. of every complete century, if divisible by 400, should also be a bissextile year, but if not so divisible, a common year.

According to this rule the year 1600 would be a bissextile year, and the years 1700, 1800, 1900 would be reckoned as common years.

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[1] When the French ambassador waited on Queen Elizabeth to report such an account of the proceedings at Paris as Charles IX. deemed proper to be announced, on entering the palace he found the apartments leading to the Queen's presence-chamber filled with crowds of courtiers in deep mourning, and as he passed through them, no one offered any friendly salutation. He said that his master had instructed him to inform the Queen that Admiral Coligny, smitten by his conscience, had confessed to the King that the Protestants had conspired to seize both him and the Queen mother, and that the dread of a civil war had driven his sovereign to allow the opposite party to proceed to the execution of their enemies. On hearing his message, the Queen expressed her esteem for King Charles, and proceeded to say :—"Although, indeed, if the information had been found true, yet the manner of cruelty used could not be allowed in any government, and least in that place, where the King might, by order of justice, have done due execution, both on the Admiral and on all others that should have been proved offenders. For it could not be denied but the same force that murdered so many multitudes might more easily have put down the leaders, especially the wounded Admiral, under arrest. She was willing to believe that those councillors, whose age and experience ought to have made them useful guides to their young sovereign, were more to be blamed than he." And the Queen concluded by saying that "she did surely persuade herself that if the King would not use his power to make some amends for so much blood, so horribly shed, God, who saw the hearts of all, as well of princes as of others, would show His justice, in time and place; when His honour should therein be glorified, as the author of all justice, and the avenger of all blood-shedding of the innocent."

By this correction of the Julian Calendar, the 14th day of the paschal moon was brought back to the same season in which it was found at the Council of Nicæa. The decree of that Council declared that Easter ought to be kept on the first Sunday after the 14th day of this moon, if this 14th day should happen on or after the 21st March. It will hence appear that according to this rule, Easter Sunday cannot happen earlier than the 22nd March nor later than the 25th April, which have been called the paschal limits.

The calendar, as corrected by Gregory XIII., is called the Gregorian Calendar, or "the New Style," and as corrected by Julius Cæsar is called the Julian Calendar, or "the Old Style." The Old Style was formally abrogated in 1582, and the New Style was adopted in all countries where the Papal supremacy was acknowledged. The Protestant States of Germany at first rejected it, as they did everything that bore any semblance of Papal authority, and did not adopt the New Style until the month of February, 1700.

In England until 1752 there were two beginnings of the year, one on the 25th March, and the other on the 1st day of January. Great inconveniences were experienced from having two modes of reckoning. At length in 1752 an Act of Parliament was passed, entitled an Act (23 Geo. II.) for regulating the commencement of the year, and for correcting the calendar. It declared that "Easter day, on which the rest depends, is always the first Sunday after the first full moon which happens upon or next after the 21st day of March; and if the full moon happens upon a Sunday, Easter day is the Sunday after." But as 170 years had elapsed since the epoch of the Gregorian correction, the Old Style had gained one additional day more on the course of the sun than it had at that epoch. By the Act it was therefore ordered that eleven days should be dropped. And accordingly, on the 2nd day of September, 1752, the "Old Style" ceased, and the next day, instead of the 3rd, was called the 14th September of the "New Style." By the same Act the beginning of the year was changed from the 25th March to the first day of January, and its reception encountered much opposition through the prejudices of people of all ranks—rich and poor, high and low;[1] and these feelings in no small degree arose from the fact of the change having originated with the Bishop of Rome. The calendar, so corrected, has ever since 1757 been printed at the beginning of the Book of Common Prayer.

The method of intercalation used in the Gregorian Calendar is not the most accurate, as 97 or 100 — 3 days are inserted in the space of four centuries. This supposes the tropical year to consist of 365 days, 5 hours, 49 min., 12 sec.: but the reformers of the calendar made use of the Copernican year, 365 days, 5 hours, 49 minutes, 20 seconds. Recent observations have determined the length of the tropical year to be 365 days, 5 hours, 48 min., 45½ sec., and if 41,851 days be intercalated in 172,800 years, there would be no error. As this mode of intercalation is different from that now in use, it is obvious that the Gregorian Calendar must be corrected after a certain number of years. The correction, however, will be inconsiderable for many ages, as it will amount only to a day and a half to be suppressed in the space of 5000 years.

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[1] On the adoption of the New Style in 1752, by Act of Parliament, it was ordered that the third day of September should be accounted as the fourteenth. This gave intense offence to the ignorant multitude, *some of whom believed that eleven days were thus taken from their lives against their will by Act of Parliament.* Hogarth, the celebrated painter, to ridicule this absurdity, represented them in one of his pictures, vociferating "Give us back our eleven days."

The modern Hebrew Calendar was settled by Rabbi Hillel, about the middle of the fourth century of the Christian era, and is founded on the periods of the moon and the sun. It is so arranged that the festival of the Passover should be celebrated on the day of the new moon at the vernal equinox, or on the day nearest to the day of the moon's conjunction with the sun. The months are arranged, some of 29 days, and others of 30 days. The intercalary year had an additional month, called Ve-Adar. This intercalary month is placed between the months Adar and Nisan, and consists of 29 days in common years and of 30 days in intercalary years. According to the scheme of Rabbi Hillel, a common year may consist of 353, 354, or 355 days, and an intercalary year of 383, 384, or 385 days; but his rules and limitations secure the reckonings of the calendar from any inconvenient discrepancy with the seasons.

The most recent scheme for the reform of the calendar was made at the French Revolution, and lasted about fourteen years. The first fruits of this Revolution were the constitution of the National Convention, the deposition of the king in 1792, and France declared to be a Republic. These proceedings were followed by the murder of the king and the queen at Paris in the following year. The Convention of the Republic, by two decrees issued on 5th Oct. and on 24th Nov., 1793, abolished the old calendar, which reckoned from the epoch of the birth of Christ, and decreed a new calendar, on what they considered true philosophical principles. But from the inconveniences arising out of the regularity of the motion of the earth on its axis and in its orbit round the sun, and of the moon round the earth, they could only change names, and decree minor subdivisions of time, and begin the year at a different epoch. Accordingly it was decreed that the era of the Revolution, 22nd September, 1792, should be distinguished as the first day of the year of the French Republic, being the day of the autumnal equinox.

The year was divided into 12 months, of 30 days each, with five complementary days. The leap years were styled Olympic years. The following descriptive epithets they gave to their new months are not new names, but merely an imitation of the old Dutch names.

Vindémiare, vintage month,	began	22 Sept.	Germinal, budding month,	began	21 Mar.
Brumaire, foggy	,,	22 Oct.	Floreal, flowery	,,	20 April
Frimaire, sleety	,,	21 Nov.	Prarial, meadow	,,	20 May
Nivôse, snowy	,,	21 Dec.	Messidor, harvest	,,	19 June
Pluviôse, rainy	,,	20 Jan.	Thermidor, hot	,,	19 July
Ventôse, windy	,,	19 Feb.	Fructidor, fruit	,,	18 Aug.

The new calendar appointed the five complementary days of the ordinary years to be celebrated as festivals: the 17th September dedicated to Virtue, the 18th to Genius, the 19th to Labour, the 20th to Opinion, and the 21st to Rewards.

In every Olympic year from 11 Ventôse (29th Feb.) to the end of the year, each day of the month was one day earlier than in ordinary years, and there were six instead of five festival days. It is curious to remark that the designation of *sans-culottides* (taken from *sans-culottes*) was given to these festival days. The name at first had been applied in contempt to the Republican party, but the same party thus attempted to invest it with honour and respect.

Each month of 30 days was divided into three decades, or periods of ten days each. The week of seven days was abolished, and the seventh day as a day of rest was no longer to be observed.

The names assigned to each of the ten days were respectively Primidi (from *primus dies*), Duodi, Tridi, Quartidi, Quintidi, Sextidi, Septidi, Octidi, Novidi, Decadi. They made the day to begin at midnight, and retained the division of the day into twenty-four hours; but the hour they divided and subdivided decimally. It appears a singular inconsistency that the Convention did not divide the day into ten parts, and the year into ten months, as they had divided the month into three parts of ten days each. It is not improbable that the Convention foresaw, if they ordained the denary division of the day and of the night, there would be found the insuperable difficulty of compelling the clocks of Paris and of the rest of France to obey the new order, to strike the hours up to ten, when they had been accustomed to strike them up to twelve, from the first day they were set up to tell the hours.

During the period of the first Republic all events and important facts connected with the history of France were recorded according to the new calendar; and to complete their work, the Archbishop of Paris and his clergy were compelled to abjure Christianity, and the Convention decreed the worship of REASON. Before the end of 1793, during the Reign of Terror, nearly 2,000 of the clergy lost their lives at the hands of the Convention. The new French Calendar continued in use until 1806, nearly two years after Napoleon Buonaparte had been proclaimed the first Emperor of the French in 1804. He then ordered the discontinuance of the new calendar, and the restoration of the Gregorian. The week of seven days was resumed, but the Christian sabbath, as a day of rest from ordinary work, has not even at the present day recovered its sanctity in Paris.

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LOGARITHMS.

In the lives of scientific men there is commonly little of interest besides their discoveries and improvements in science, and their writings and correspondence with the learned of their own times. The inventor of logarithms, however, was eminent in other respects than as a man of science. His patriotic conduct and decided Christian character, exhibited on difficult occasions, are not unworthy of some brief notices in connection with his important inventions in the mathematical sciences. John Napier, son of Sir Archibald Napier, of Merchiston, master of the mint in Scotland, was born in the year 1550. His mother's brother was the first reformed bishop of Orkney, and the wise advice respecting the early education of his nephew appears not to have been tendered in vain.

In the fourteenth year of his age, young Napier was incorporated a member of St. Salvator's College in St. Andrew's University, which only a few years before had been almost deserted in consequence of the tumults raised against the Reformation. About that time, 1563, it appears that the students of St. Andrew's were exercised once a week in theological disputations, at which one of the masters presided. The students were exhorted to avoid the altercations usually practised in the schools, and to behave themselves as men desirous of mutual instruction, and as the servants of Christ, who ought not to strive, but to be gentle to all. Young Napier had before studied the sacred Scriptures, and aspiring to become a decided Protestant, he applied his energies to the sacred cause of truth.

This fact is derived from his own words in the address "To the Godly and Christian Reader," prefixed to his "Plain Discovery of the Revelation of St. John," published in English in 1593.[1] He writes: "In my tender yeares and barneage in Sanct-Androis, at the schooles, having, on the one part, contracted a loving familiaritie with a certaine gentleman, &c., a Papist; and, on the other part, being attentive to the sermons of that worthie man of God, Maister Christopher Goodman, teaching upon the Apocalyps, I was so mooved in admiration against the blindnes of Papists, that could not most evidently see their seven-hilled citie, Rome, painted out there so lively by Saint John as the mother of all spirituall whoredome, that not onely bursted I out in continual reasoning against my said familiar, but also from thenceforth I determined with myselfe (by the assistance of God's Spirit) to employ my studie and diligence to search out the remanent mysteries of that

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[1] His work on the Apocalypse contains much that is sound, and much that is plausible; but, like all who have ventured to open the seals of unfulfilled prophecy, he has failed to convince any rational man that he has been admitted to the secrets of the kingdom of heaven. His study of the prophetic visions of St. John appears to have led him to assume the prophetic character (always a dangerous assumption), and to predict that the day of judgment was to happen between the years 1688 and 1700 (Book I., Prop. 14). It is, perhaps, almost needless to remark that the progress of time has proved that Napier, as well as a later Scotch prophet, the Rev. Edward Irving, had both made a mistake in converting their conjectures into prophetic announcements.

Holy Book; as to this houre (praised be the Lorde), I have bin doing at al such times as conveniently I might have occasion."

It is not known how long he remained a student at St. Andrew's; it is certain he had left before 1588, when his contemporaries were admitted to the degree of Master of Arts. It is probable that under the advice of his uncle he proceeded to the eminent schools of learning in the Low Countries, and to the University of Paris, where he had the advantage of hearing the most distinguished men of that age, and among them Peter Ramus, the eminent mathematician, who was one of the victims at the massacre of St. Bartholomew. It is not known whether the troubles in his own country, or the state of affairs on the continent, were the causes which led to the return of John Napier to his home, probably in 1571, as it appears that his first marriage took place in 1572, and by his first wife he had one son and one daughter. Some time after the death of his first wife he married again, and had a second family of five sons and four daughters. Some of his sons became distinguished, and such were his daughters, that it is reported of them, "that they were all blessed with honourable or respectable marriages."

John Napier, firmly devoted to the Protestant cause, was ever loyal to his sovereign. He lived apart from the intrigues and plots of the Papists, though connected with them by his second marriage; and in the midst of his mathematical lucubrations and theological studies, was always ready to take his part in public duty. In 1584 King James made his first determined attack on the privileges of the Church, and his duplicity and inconsistent conduct caused great troubles to the Church while beset by powerful enemies from abroad. In the memorable year 1588 Napier was chosen one of the Commissioners to the General Assembly called together at Edinburgh under circumstances the most alarming for Church and State. On this occasion his mind seems to have been much agitated, as will appear from the following extract from the preface to his "Plain Discovery":—

"This new insolence of Papists, arising about the 1588 year of God, and dayly incresing within this iland, doth so pitie our hearts, seeing them put more trust in Jesuites and Seminarie priests, than in the true Scriptures of God; and in the Pope and King of Spaine, than in the King of kings; that to prevent the same, I was constrained of compassion (leaving the Latin) to haste out in English this present worke, almost unripe, that thereby the simple of this iland may be instructed, the godly confirmed, and the proud and foolish expectations of the wicked beaten downe."

The destruction of the Invincible Armada had not entirely discouraged the King of Spain in his designs against Britain. For in the year after, the Duke of Parma supplied a large sum of money to the Papist party in Scotland, and a plot was contemplated that 30,000 men from Spain should land on the west coast of Scotland, march to Carlisle, and invade England, leaving 5,000 Spaniards with the leaders in Scotland. The plot was frustrated when it was nearly ripe for execution.

After this event Napier was sent as the chief of a mission to deliver to the King in person a petition from the General Assembly for the punishment of the rebels, the safety of the Church, and the quieting of the public mind.

It is remarked by his chief biographer, when Napier and the delegates were ushered into the royal presence: "It must have been a scene worthy of historical painting—this interview between the grotesque King of Scotland and the recluse philosopher. We may imagine the monarch, as portrayed in that ancient description of him which seems to have been drawn by an actual observer. 'Of a middle stature, more corpulent through his clothes than in his body, yet fatt enouch, his clothes ever being made large and easie, the doublets quilted for steletto proofe, his breeches in grate pleits, and full stuffed; of a timorous disposition, which was the greatest reasone of his quilted doublets; his eyes large, ever roulling after aney stranger cam in his presence; in so much as maney, for shame, have left the roome as being out of countenance: his beard werey thin; his toung too large for his mouth, &c.,'—confronted with John Napier, with his serene presence, thoughtful eye, and ample beard, rarely seen within the royal circle."

On the 29th January, 1593, John Napier wrote an epistle[1] to King James against his collusively favouring the Papists, and urged reforms both in Church and State. This epistle he also prefixed as the dedication to that monarch of "The Plain Discovery," which was published the same year. It is described by one of his biographers as containing "remonstrance without sedition, rebuke without disloyalty, and admonition without impertinence." "The Plain Discovery" was translated into French, and three editions were published at Rochelle, the first in 1602.

After the detection of the Spanish plot, Napier was engaged in the invention of some plans and machines for the defence of the island, which were communicated by King James's ambassadors to the English Government. A description of them is preserved in his handwriting, and bearing his signature. The paper is prefaced by the words: "Anno Domini 1596, the 7 June, Secrett Inventionis, profitabill and necessary in theis dayes for defence of this Iland, and withstanding of strangers, enemies of God's truth and religion."

How John Napier was led to the invention of logarithms appears from his own account of the matter. He had made several improvements in trigonometry: two of the most important have since been connected with his name—"Napier's Rules," and "Napier's Analogies." He became desirous of finding out some method by which he could abridge the labour of numerical computations connected with this subject. In the year 1614, when he was above sixty years of age, he published his discovery at Edinburgh in a work entitled, "Mirifici Logarithmorum Canonis Descriptio," reserving his method of constructing the tables until he knew what the learned might think of his invention. The work is dedicated to Prince Charles, the son of King James. The dedication opens with the following expressive sentiment:—

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[1] The following is a short extract from the epistle:—"Praying your Majesty to attend yourself unto these enormities, and (without casting over the credite thereof to wrong wresters of justice), your Majesty's self to wit, certainly that justice be done to these your true and godly lieges, against the enemies of God's church, and their most cruel oppressors. Assuming your Majesty be concordance of al Scriptures, that if your Majesty ministrate justice to them, God the supreme judge shall ministrate justice to you against al your enemies, and contrarily, if otherwise. Therefore, sir, let it be your Majesty's continual study (as called and charged thereunto by God) to reform the universall enormities of your country, and to begin at your Majesty's owne house, family, and court, and purge the same of all suspicion of Papists and Atheists, or Newtrals, whereof this Revelation foretelleth that the number shall greatly increase in these latter daies. For shall any prince be able to be one of the destroyers of that great secte, and a purger of the world from Antichristianisme, who purgeth not his owne country? shall he purge his whole country who purgeth not his own house? or shall he purge his house, who is not purged himselfe?"

"Seeing there is neither study nor any kind of learning that doth more actuate and stir up generous and heroical wits to excellent and eminent affairs; and contrariwise that doth more deject and keep down sottish and dull minds than the mathematics; it is no marvel that learned and magnanimous princes in all former ages have taken great delight in them, and that unskilful and slothful men have always pursued them with most cruel hatred, as utter enemies to their ignorance and sluggishness." And further adds, "And therefore this invention (I hope) will be much the more acceptable to your Highness, as it yieldeth a more easy and speedy way of accompt. For what can be more delightful and more excellent in any kind of learning than to despatch honourable and profound matters exactly, readily, and without loss of either time or labour?"

The preface of the work is interesting, being addressed to students of the mathematics. The following copy is taken from the English translation of Napier's work, as it contains some additions made by the author himself:—

"Seeing there is nothing (right well beloved students in the mathematickes) that is so troublesome to mathematicall practise, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which, besides the tedious expense of time, are for the most part subject to many slippery errors. I began, therefore, to consider in my minde, by what certaine and ready art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent briefe rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this, which, together with the hard and tedious multiplications, divisions, and extractions of rootes, doth also cast away from the worke itselfe, even the very numbers themselves that are to be multiplied, divided, and resolved into rootes, and putteth other numbers in their place, which perform as much as they can do, only by addition and subtraction, division by two or division by three; which secret invention, being (as all other good things are) so much the better as it shall be the more common; I thought good heretofore to set forth in Latine for the publique use of mathematicians. But now some of our countrymen in this island well affected to these studies, and the more publique good, procured a most learned mathematician to translate the same into our vulgar English tongue, who after he had finished it sent the coppy of it to me, to be seene and considered on by myself.

"I having most willingly and gladly done the same, finde it to be most exact and precisely conformable to my minde and the originall. Therefore it may please you who are inclined to these studies, to receive it from the translator with as much good will as we recommend it unto you. Fare ye well."

This work contains the natural sines and the logarithms of the sines for every minute of the quadrant, with a description and explanation of the uses of the tables. A translation[1] of this work[2] was made into English by Edward Wright, M.A., Fellow of Gonville and Caius College, Cambridge, who died in 1615. The translation, however, was published in 1618 by his son Samuel Wright (then a scholar of the same college), and dedicated to the East India Company.

A preface to the reader was added by Henry Briggs, and "a table to finde the part proportionall." Henry Briggs was educated at St. John's College, Cambridge; admitted to the degree of M.A. in 1585; elected Fellow in 1588, and appointed reader on Dr. Linacre's foundation in 1592. In 1596 he was chosen the first reader in Geometry in Gresham College, London, and afterwards, in 1619, he was appointed the first Savilian professor of geometry at Oxford. Soon after the publication of the "Canon Mirificus," Briggs communicated in his lectures at Gresham College the improvement of making 1 the logarithm of the ratio of 10 to 1, instead of 2·30258, as Napier had done. And from the evidence that exists, it appears that he was the first person who publicly made known the advantages of this change in the scale, which he also communicated to Napier himself.

In a letter to Mr. (afterwards Archbishop) Usher, of the date of 10th March, 1615, he writes: "Napier, lord of Markinston, hath set my head and hands at work with his new and admirable logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better and made me more wonder. I purpose to discourse with him concerning eclipses, for what is there we may not hope for at his hands." Accordingly he visited Napier in the summer of 1615. Their first meeting was described to William Lilly, who has thus recorded the account in his "Life and Times."

"I will acquaint you with one memorable story related unto me by John Marr, an excellent mathematician and geometrician, whom I conceive you remember. He was servant to King James I. and Charles I. At first, when the Lord Napier, or Marchiston, made public his logarithms, Mr. Briggs, then reader of the Astronomy Lectures at Gresham College, in London, was so surprised with admiration of them, that he could have no quietness in himself until he had seen that noble person the Lord Marchiston, whose only invention they were. He acquaints John Marr herewith, who went into Scotland before Mr. Briggs, purposely to be there when these two learned persons should meet. Mr. Briggs appoints a certain day when to meet at Edinburgh; but failing thereof, the Lord Napier was doubtful he would not come. It happened one day, as John Marr and the Lord Napier were speaking of Mr. Briggs, 'Ah, John (said Marchiston), Mr. Briggs will not now come.' At the very instant one knocks at the gate; John Marr hasted down, and it proved Mr. Briggs, to his great contentment. He brings Mr. Briggs up into my lord's chamber, where almost one quarter of an hour was spent, each beholding the other almost with admiration before one word was spoken. At last Mr. Briggs began: 'My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, viz., the logarithms; but, my lord, being by you found out, I wonder nobody else found it out before, when now known it is so easy.'"

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[1] The following "Admonition" does not appear in the original Latin, page 22, sect. 9, cap. 4. It was most probably added by Napier himself when he revised Wright's translation. It refers to that system of the logarithms of the natural numbers in which 1 is made the logarithm of the ratio of 10 to 1. "An Admonition. But because the addition and subtraction of these former numbers may seem somewhat painful, I intend (if it shall please God) in a second edition, to set out such logarithms as shall make those numbers above written to fall upon decimal numbers, such as 100,000,000, 200,000,000, 300,000,000, &c., which are easie to be added or abated to or from any other number."

[2] The complete title-page is:—"A description of the admirable Table of Logarithmes:—with a declaration of the most Plentifull, Easie, and Speedy use thereof in both kinds of Trigonometry, as also in all Mathematical Calculations. Invented and published in Latine by that Honourable Lord John Nepair, Baron of Marchiston, and translated into English by the late learned and famous Mathematician, Edward Wright. With an addition of the Instrumentall Table to finde the part Proportionall, intended by the Translator, and described in the end of the Booke, by Henrie Brigs, Geometry-reader at Gresham House in London. All perused and approved by the Authour, and published since the death of the Translator. Whereunto is added new Rules for the ease of the student. London, printed for Simon Waterson, 1618."

In the year 1617, shortly before his death, Napier published his "Rabdologia." In the dedication is the following passage:—"The difficulty and prolixity of calculation, the weariness of which is so apt to deter from the study of mathematics, I have always, with what powers and little genius I possess, laboured to eradicate. And with that end in view I published of late years the Canon of Logarithms wrought out by myself a long time ago, which, casting aside the natural numbers, and the more difficult operations performed by them, substituting in their place others affording the same results, by means of easy additions, subtractions, bisections, and trisections. Of which logarithms, indeed, I have now found out another species much superior to the former, and intend, if God shall grant me longer life, and the possession of health, to make known the method of constructing as well as the manner of using them. But the actual computation of this new canon I have left, on account of the infirmity of my bodily health, to those versant in those studies; and especially to that truly most learned man, Henry Briggs, public Professor of Geometry in London, my most beloved friend."

In the following year Briggs paid a second visit to Napier, and after his return to London printed, in 1617, his "Chilias Prima Logarithmorum," but did not publish it till the next year after the death of Napier, which happened on the 3rd April, 1618; for in his preface Briggs writes, "Why these logarithms differ from those set forth by their most illustrious inventor, of ever respectful memory, in his 'Canon Mirificus,' it is to be hoped his posthumous work will shortly make appear."

The posthumous work was published by his son, Robert Napier, in 1619, with the title of "Mirifici Logarithmorum Canonis Constructio." In the preface, speaking of his father, he writes:—"You have then (benevolent reader) the doctrine of the construction of logarithms—which here he calls artificial numbers, for he had this treatise beside him composed for several years before he invented the word logarithm [λόγων ἀριθμός]—most copiously unfolded, and their nature, accidences, and various adaptations to their natural numbers perspicuously demonstrated. I have also thought good to subjoin to the construction itself a certain appendix, concerning the method of forming another and more excellent species of logarithms, to which the inventor alludes in his epistle prefixed to the 'Rabdologia,' and in which the logarithm of 1 is 0.[1] I have also published some lucubrations upon the new species of logarithms, by that most excellent mathematician, Henry Briggs, public professor in London, who undertook most willingly the very severe labour of calculating this canon, in consequence of the singular affection that existed between him and my father of illustrious memory." Robert Napier in this volume makes no allusion to the claim of Briggs, as his father, in the "Rabdologia," laid claim to that improvement, and stated that he had committed the execution of it to Briggs. From an expression in Briggs's preface to his "Chilias Prima Logarithmorum," it would appear he expected a recognition of his claim in the posthumous work of Napier. But as it had been passed over in silence, Briggs, in the preface to his "Arithmetica Logarithmica," clearly declared the part he had taken, and that he had first suggested the improvement in his lectures.

In the year 1624 Briggs published his great work "Arithmetica Logarithmica," of which a translation in English appeared in 1631. In the address to his readers he gives the following account of the part he took in the improvement of logarithms, and his great labour in the calculation of the improved tables. "Be not surprised that these logarithms are different from those which that illustrious man, the Baron of Marchiston, published in his 'Canon Mirificus.' For when explaining publicly the doctrine of them to my auditors at Gresham College, in London, I remarked that it would be much more convenient that 0 should stand for the logarithm of the whole sine [or radius] as in the 'Canon Mirificus,' but that the logarithm of the tenth part of the same whole sine, namely, of 5° 44′ 21″, should be 10,000,000,000. And concerning this matter I immediately wrote to the author himself; and as soon as the season of the year, and my public teaching would permit, I went to Edinburgh, where being most kindly received by him, I staid a whole month. But when we began to converse about this change in the system, he said that for some time [dudum] he had been sensible of the same thing, and had desired to accomplish it, but however he had published those that he had already prepared, until he could make others more convenient if his duties and feeble health would permit. But he thought that the change ought to be effected in this way—that 0 should be made the logarithm of 1; and 10,000,000,000 the logarithm of the whole sine, which I could not but acknowledge was by far the most convenient of all. Therefore, rejecting those which I had before prepared, I began at his exhortation to ponder seriously about the calculation of these tables, and in the following summer I went again to Edinburgh, and showed him the principal part of those tables which are here published, and I was about to do the same the third summer, if it had pleased God to spare him so long."

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[1] In this appendix he shows how the logarithms of all composite numbers can be found from the logarithms of prime numbers, and thus describes his method. "In order to find the logarithms of all numbers, it is necessary that the logarithms of some two natural numbers be given, or at least assumed, as in the former first construction 0, or cipher, was assumed for the logarithm of the natural number 1, and 10,000,000,000 for the logarithm of the natural number 10. These, therefore, being given, the logarithm of the natural number 5 (which is a prime number) is sought in this manner. Between 10 and 1 is sought the geometric mean, which is $\frac{316227766017}{1000000000000}$. So between 10,000,000,000 and 0 is sought the arithmetic mean, which is 5,000,000,000. Next between 10 and $\frac{316227766017}{1000000000000}$ is taken the geometric mean, which is $\frac{562341325191}{1000000000000}$. And similarly between 5,000,000,000 and 10 is taken the arithmetic mean, which is 75,000,000,000." It will be seen by this process that the successive arithmetic means are the logarithms of the corresponding geometric means. But as these geometric means are not the natural prime numbers, if the process be continued it will be found, after twenty-five operations, that the geometric mean (taking seven places of figures) will be very nearly equal to 9, being defective only by the five millionth part of an unit; and the corresponding arithmetic mean may be taken without sensible error as the logarithm of 9. Thus the logarithm of 9 being known, the logarithm of the prime number 3 is also known.

This work[1] contains the logarithms (1 being the logarithm of 10, and 0 of 1) of numbers from 1 to 20,000, and from 90,000 to 100,000, all to 15 places of figures, with a method of finding the logarithms of the intermediate numbers. It may be noted that in some copies of the "Arithmetica Logarithmica" there is found a table of the logarithms of the numbers 100,000 to 101,000, which appears to have been added after the former tables had been printed. From the time of explaining the "Canon Mirificus" in his lectures till the publication of his great work, he always regarded Napier as his guide. Both the character of Napier, his friendly intercourse with Briggs, and the respect with which they both wrote of each other, are scarcely in unison with the opinion expressed by Dr. Hutton in the introduction to his "Tables of Logarithms."

"Upon the whole matter, it seems evident that Briggs, whether he had thought of this improvement in the construction of logarithms, of making 1 the logarithm of the ratio of 10 to 1, before Lord Napier or not (which is a secret that could be known only to Napier himself), was the first person who communicated the idea of such an improvement to the world; and that he did this in his lectures to his auditors at Gresham College in the year 1615, very soon after his perusal of Napier's 'Canon Mirificus Logarithmorum' in the year 1614. He also mentioned it to Napier, both by letter in the same year, and on his first visit to him in Scotland, in the summer of the year 1616, when Napier approved the idea, and said it had already occurred to himself, and that he had determined to adopt it. It would therefore have been more candid in Lord Napier to have told the world, in the second edition of this book, that Mr. Briggs had mentioned this improvement to him, and that he had thereby been confirmed in the resolution he had already taken, before Mr. Briggs's communication with him (if, indeed, that was the fact), to adopt it in that his second edition, as being better fitted to the decimal notation of arithmetic which was in general use. Such a declaration would have been but an act of justice to Mr. Briggs; and the not having made it, cannot but incline us to suspect that Lord Napier was desirous that the world should ascribe to him alone the merit of this very useful improvement of the logarithms, as well as that of having originally invented them; though, if the having first communicated an invention to the world be sufficient to entitle a man to the honour of having invented it, Mr. Briggs had the better title to be called the first inventor of this happy improvement of logarithms."

The same kindly feelings existed between Briggs and Robert

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[1] The title-page of the work bears these words:—"Hos numeros primus invenit Clarissimus Vir Johannes Neperus, Baro Merchistonii; eos autem ex ejusdem sententia mutavit, eorumque ortum et usum illustravit Henricus Briggius, in Celeberrima Academia Oxoniensi, Geometriæ Professor Savilianus."

Napier after his father's death, and it appears that a copy of Napier's book on Arithmetic was copied out by Robert Napier for Mr. Henry Briggs, the Savilian Professor at Oxford. Briggs never made any claim to be the inventor of logarithms; and the fact of the first mention of his improvement publicly in his lectures is quite consistent with Napier's statement made at his first interview with Briggs, that he had before been aware of the advantages of the method Briggs had proposed.

Briggs had also completed before his death, which happened in 1630, a table of the logarithms of the sines, tangents, and secants to fifteen places of figures, and annexed it to his table of the natural sines, tangents, and secants which he had before calculated. This work he committed to his friend Henry Gellibrand, then professor of geometry in Gresham College, who published it in 1633 with the title of "Trigonometria Britannica." These tables for their accuracy have been seldom found to differ from the truth by more than a few units in the fifteenth figure.

No one before Napier ever considered all numbers as expressions of proportions, which could be included in a series of ratios. This idea is the basis of his invention, and the effect was, that he devised a method of finding a series of proportionals, containing all numbers, and every number having its own exponent, and also a method of finding the *exponent* of any given number of the series, or the *number* of any given exponent.

The merit of Napier consists of having imagined and assigned a logarithm to any number whatever, by supposing the logarithm of that number to be one of the terms of a series in arithmetical progression, and the number itself one of the terms of a geometrical progression whose successive terms differ by very small increments from each other.

The natural numbers, 1, 2, 3, 4, 5, &c., form an arithmetical progression, and may be made the logarithms of a series of numbers in geometrical progression: and the same numbers, 1, 2, 3, &c., may be also made to represent the logarithms of different geometrical series. But the natural numbers, of which the logarithms are wanted, form of themselves an arithmetical series, and can never become a geometrical series. Here arose the difficulty, how could one series of numbers in arithmetical progression be made to correspond with the whole series of the natural numbers, in their order of increase, which are themselves also a series of numbers in arithmetical progression?

The arithmetical progression, 0, 1, 2, 3, 4, 5, &c., corresponds with the geometrical progression, 1, 2, 4, 8, 16, 32, &c., and the numbers in order of the former can be made the logarithms of the corresponding numbers of the latter series; thus, 0 may be made the logarithm of 1, 1 of 2, 2 of 4, 3 of 8, 4 of 16, 5 of 32, &c., and so on; but here arose the difficulty, how could the logarithms of the numbers intermediate be represented? the logarithm of 3 lying between 2 and 4; of 5, 6, 7, between 4 and 8; of 9, 10, 11, 12, 13, 14, 15, between 8 and 16, &c.; so as to be made to correspond to all numbers in their natural order of increase, these natural numbers themselves being in arithmetical progression. Napier had not the aid of the algebraical notations afterwards devised, and he knew nothing of the equation $a^x = y$, nor had he even conceived the idea of the base of a system of logarithms.[1] He was led to his invention solely by geometrical considerations. Napier's object was not to calculate the logarithms of the natural numbers, but to facilitate calculations in trigonometry. This is the reason of his computing only the logarithms of the sines, and the logarithms of the complements of the sines, (the word *cosine* not yet having been devised,) from which the logarithms of the tangents, &c., could be easily deduced. The conception he formed was that of two points, generating straight lines by very small increments, and so regulated that in one case the successive increments should be equal; and in the other, that they should differ proportionally from each other in an indefinitely small degree.[2] He knew that the indefinitely small ratios which he imagined to be generated between the natural numbers were not exact, but only approximations, and the excess or defect would become less than any quantity he might wish to assign, and therefore would not materially affect the results of his calculations, so far as he intended them to apply.

As the arcs of a circle were considered as the measures of the angles subtended at the center, and the sines of all arcs as parts of the radius, which was the sine of the whole quadrant, he conceived a line *az* equal to the radius, and a point beginning to move from *a*, so that in equal times it moved over successive parts *ab*, *bc*, *cd*, &c., of the line *az* in a decreasing geometrical progression, leaving the successive remainders, *bz*, *cz*, *dz*, &c., also in geometrical progression.

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[1] In Dr. Booth's Treatise on some new Geometrical Methods, vol. i., ch. 32, on the geometrical origin of logarithms, he has exhibited the value of the Napierian base in terms of the functions of the angle which a focal perpendicular on a tangent makes with the axis of a parabola. He remarks:—"We are thus (for the first time, it is believed) put in possession of the geometrical origin of that quantity so familiarly known to mathematicians—the Napierian base." The value of the angle is 49° 36′ 18″.

[2] The first two chapters of the Canon Mirificus afford a view of Napier's method. The following are his Definitions and Propositions, but without his illustrations, taken verbally from the translation of Edward Wright:—

CHAP. I. Of the Definitions:

Def. 1. A line is said to increase equally, when the poynt describing the same goeth forward equall spaces, in equall times, or moments.

Cor. Therefore, by this increasing, quantities equally differing, must needes be produced, in times equally differing.

Def. 2. A line is said to decrease proportionally into a shorter, when the poynt describing the same in equall times, cutteth off parts continually of the same proportion to the lines from which they are cut off.

Cor. Hence it followeth that by the decrease in equall moments (or times) there must needes also be left proportional lines of the same proportion.

Def. 3. Surd quantities, or unexplicable by number, are said to be defined, or expressed by numbers very neere, when they are defined or expressed by great numbers which differ not so much as one unite from the true value of the surd quantities.

Def. 4. Equal-timed motions are those which are made together, and in the same time.

Def. 5. Seeing that there may be a slower and a swifter motion given than any motion, it shall necessarily follow, that there may be a motion given of equall swiftnesse to any motion (which we define to be neither swifter nor slower).

Def. 6. The logarithme therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the

On another line, AZ, not definite as the former, he conceived a point beginning from A to move over equal parts, AB, BC, CD, &c., of this line, with the velocity uniform, the same as the initial velocity of the other point moving in az. Now at the end of the first, second, third, &c., equal portions of time, the moving point in az is found at b, c, d, &c.; and that in AZ at B, C, D, &c., respectively. The lines za, zb, zc, &c., will be a series of lines in a *decreasing* geometrical progression, and o, AB, AC, AD, &c., will be a corresponding series of lines in an *increasing* arithmetical progression. The lines o, AB, AC, AD, &c., may be made the logarithms of the series of lines za, zb, zc, &c., supposing the point in az moving with a velocity decreasing in proportion to its distance from z, while the velocity of the point in AZ moves with the same constant velocity as it had at the beginning of its motion.

As two independent conditions are necessary to limit every system of logarithms, as if 0 be taken for the logarithm of 1, and any definite number be assumed for the logarithm of some other number, a system of logarithms can be computed under these conditions. Napier, however, assumed the logarithm of the whole sine to be 0, and hence as the series of the logarithms of the sines *increase*, the sines themselves *decrease*. He also assumed that the points beginning to move from a and A in the lines az and AZ with equal velocities, the increments described in the first small portions of time are equal, or that the natural sines and their logarithms near the whole sine have equal differences, but different affections. With these limitations, there is explained in his posthumous work his methods of computing his canon of logarithms, which at first he styled artificials, or artificial numbers, to distinguish them from the numbers which denoted the natural sines.

There is a singular coincidence of ideas exhibited in Napier's invention of logarithms, and in Sir Isaac Newton's invention of fluxions, as will appear from the following account by Newton him-

line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.

A Consequent. Therefore the logarithme of the whole sine 1000000 is nothing or 0; and consequently the logarithmes of numbers greater than the whole sine, are lesse than nothing.

Therefore we call the logarithmes of the sines, *Abounding*, because they are alwayes greater than nothing, and set this mark + before them, or else none. But the logarithmes which are less than nothing, we call *Defective*, or wanting, setting this mark − before them.

Chap. II. Of the Propositions of Logarithmes:

Prop. 1. The logarithmes of proportionall numbers and quantities are equally differing.

Prop. 2. Of the logarithmes of three proportionals, the double of the second or meane, made lesse by the first, is equall to the third.

Prop. 3. Of the logarithmes of three proportionals, the double of the second, or middle one, is equall to the summe of the extremes.

Prop. 4. Of the logarithmes of foure proportionals, the summe of the second and third, made lesse by the first, is equal to the fourth.

Prop. 5. Of the logarithmes of foure proportionals, the summe of the middle ones, that is, of the second and third, is equall to the logarithme of the extreames, that is to say, the first and fourth.

Prop. 6. Of the logarithmes of foure continuall proportionals, the triple of either of the middle ones is equall to the summe of the further extreame, and the double of the neerer.

self:—"I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described, and therefore generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux; and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles. Therefore, considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated, I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments *Fluxions*, and the generated quantities *Fluents*, I fell by degrees, in the years 1655 and 1656, upon the method of fluxions, which I have made use of here in the quadrature of curves." [1]

Newton wrote his paper on Fluxions in 1665, the year before he took his B.A. degree. It appears that the minds both of Napier and Newton were led independently by the same ideas to effect the objects they proposed; one to construct a table of logarithms, the other to devise a calculus for dealing with variable magnitudes.

Edmund Gunter was appointed Professor of Astronomy at Gresham College in 1619, while Henry Briggs was there the Professor of Geometry, and he held the office till his death, which happened in 1626. In 1620 he published his "Canon of Triangles," which contains the logarithmic sines and tangents for every minute of the quadrant, calculated to seven places of figures. In the year 1623 he reprinted these tables in his work "De Sectore et radio," and added the "Chilias Prima" of his colleague Briggs. He introduced the use of the arithmetical complement into the arithmetic of logarithms, and was the first who employed the word *cosine* for the sine of the complement of an arc. In 1623 he also applied the logarithms of numbers, of sines and of tangents, to straight lines divided on a scale by which proportions in numbers and in trigonometry could be resolved by means of a pair of compasses. His method of division was founded on the property, that the logarithms of the terms of equal ratios are equidifferent. This instrument, in the form of a two-foot scale, has long been in use for navigation and other purposes, and is known by the name of "Gunter's Scale." The logarithmic lines on Gunter's scale were afterwards drawn in different ways. In 1627 they were drawn by Edmund Wingate on two separate scales sliding against one another, to save the trouble of using the compasses. And they were also in 1627 applied by Oughtred to concentric circles. In 1627 and 1628 Adrian Vlacq reprinted at Gouda, in Holland, the "Arithmetica Logarithmica" of Briggs, and added the logarithms which Briggs had omitted, from 20,000 to 90,000. But these he had computed only to ten places of figures. He added also a table of natural sines, tangents, and secants to every minute of the quadrant.

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[1] Quantitates Mathematicas non ut ex partibus quam minimis constantes, sed ut motu continuo descriptas hic considero. Lineæ describuntur ac describendo generantru non per appositionem partium sed per motum continuum punctorum, superficies per motum linearum, solida per motum superficierum, anguli per rotationem laterum, tempora per fluxum continuum, et sic in cæteris. Hæ geneses in rerum natura locum vere habeat et in motu corporum quotidie cernuntur. Et ad hunc modum veteres ducendo rectas mobiles in longitudinem rectarum immobilium genesin docuerunt rectangulorum.

Considerando igitur quod quantitates æqualibus temporibus crescentes et crescendo genitæ, pro velocitate majori vel minori qua crescunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum quibus generantur; et has motuum vel incrementorum velocitates nominando *Fluxiones*, et quantitates genitas nominando *Fluentes*, incidi paulatim Annis 1665 et 1666 in Methodum Fluxionum qua hic usus sum in quadratura curvarum.—*Tractatus de Quadratura Curvarum.*

Gregory St. Vincent, in his "Opus Geometricum Quadraturæ circuli et Sectionum Coni," published in 1647, shows that if one asymptote of a hyperbola be divided into parts in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide into equal portions the spaces contained between the asymptote and the curve. It was afterwards observed that by taking the continual sums of these parts, there would be obtained areas in arithmetical progression corresponding to the abscissæ in geometrical progression, and that these areas and abscissæ would be analogous to a system of logarithms and their corresponding numbers. On account of this analogy Napier's logarithms have been named hyperbolic logarithms, but this analogy is not applicable to Napier's logarithms only, but to all other possible systems of logarithms. Nor does it illustrate his idea of the generation of logarithms; Napier's exact idea, however, is completely illustrated by another curve. If the lines which represent Napier's logarithms be taken on a line as abscissæ, and lines which denote the corresponding natural numbers be drawn at right angles as ordinates to these abscissæ, the curve passing through the extremities of the ordinates will be the logarithmic curve. Its properties were first described by Huygens in his "Dissertatio de Causa Gravitatis," where it is shown that its subtangent is constant. This curve was also considered by Roger Cotes, the Plumian Professor at Cambridge, in his "Harmonia Mensurarum," which was published in 1722. He named the constant subtangent of the curve the *modulus* of the system of logarithms, being a fourth proportional to the increment of the ordinate, the increment of the abscissa, and the ordinate itself. But under Briggs's idea of logarithms, being rather numeral than geometrical, the modulus may be considered as the natural number at that point of the system where the increment of the number is equal to the increment of the logarithm, or when the consecutive numbers and their logarithms have equal differences. And it will appear that the logarithms of equal numbers in any two systems are proportional to their modulus. The ratio which connects the systems of Napier and Briggs is 2·3025850 to 1.

Napier's theory, however ingenious in itself, was felt by mathematicians to labour under the objection of treating geometrically a subject which was in its nature arithmetical, as Briggs had exhibited in the tables he had calculated and published. Their objection was founded on the definition of logarithms, "numeri rationem exponentes," and in course of time various improvements were made in the methods of constructing them—more simple, and involving less labour than the methods of Napier and Briggs. It was about fifty years after Napier's death before series were employed. The series for log. $(1+x)$ was first invented by Mercator, in 1667, and published in his "Logarithmotechnica." James Gregory, in his "Exercitationes Geometricæ," proved the series for log. $(1+x)$, log. $(1-x)$, and log. $(1+x)$—log. $(1-x)$.

Dr. Edmund Halley, in a paper printed in the "Transactions of the Royal Society for 1595," showed how to find the logarithms of numbers, and to solve the converse problem by a process of algebra. This paper ends with these words:—"Thus I hope I have cleared up the doctrine of logarithms, and shown their construction and use independent from the hyperbola, whose affections have hitherto been made use of for this purpose, though this be a matter purely arithmetical, nor properly demonstrable from the principles of geometry. Nor have I been obliged to have recourse to the method of indivisibles, or the arithmetic of infinites, the whole being no other than an easy corollary to Mr. [Sir Isaac] Newton's 'General Theorem' for forming roots and powers."

Euler, in his "Introductio in Analysin Infinitorum," has exhibited the properties of logarithms and the series for the construction of tables nearly in the form in which they are employed by mathematicians at the present time.

For the details of the history and the various methods and improvements, the student is referred to the works of the original authors, which he will find noted in the "Scriptores Logarithmici" of Baron Maseres, and in the introduction prefixed to the "Tables of Logarithms" published by Dr. Hutton.

ELEMENTARY ARITHMETIC,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION VI.

INTEGERS, ABSTRACT.

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NUMERATION AND NOTATION.

Art. 1. It is probable that the earliest efforts in counting were very limited, and did not extend beyond the simple wants of primitive life. As the human race multiplied on the earth and their wants increased, different methods of numbering would be devised, and improvements made, until necessity, the mother of inventions, led to the discovery of that perfect system which has superseded all others, and is now in almost universal use in the civilised world. To count or to number is nothing more than to form ideas of assemblages of things considered as units, and to assign names to each of these assemblages. This is the first process of the human mind in the science of numbers. Every particular object or thing suggests the idea of unity or one thing to the mind, and every assemblage of things suggests the idea of numbers composed of a greater or less assemblage of units. If such a collection of things be considerable, it is impossible to determine at first sight, with exactness, the whole number, and to name the sum of them. Our senses suggest only the undefined idea of a multitude. Nature has provided a kind of arithmetical instrument, more generally used than is commonly imagined, namely, the two hands, with the five fingers on each hand. It is highly probable that these were the first instruments used by men to assist them in the art of numbering. There is a strong presumption in favour of the truth of this opinion, that all civilised nations from the earliest times have counted by the number of the fingers on one or on both hands. No reason can be discovered why the number five or ten should be chosen rather than any other number for the basis of counting, except the primitive practice of counting by the fingers of one or of both hands. It is, besides, not improbable that in early times men counted by their fingers, objects which did not exceed that number. They could also count the exact number of tens in an assemblage of objects, and note the units which remained less than tens. But as the fingers could only serve them to ascertain that remainder of units above the tens, they wanted something to determine the number of tens. When this number was large, they were led to look for new helps. Nature presented them with many things equally fit to assist them in this operation, as pebbles or counters. It is now easy to imagine how that, by the help of the fingers and little stones, a method of numbering might be devised, and considerable calculations might be performed.

A heap of pebbles consisting of individual pebbles can be numbered by arranging them in sets of ten pebbles each, and reserving the pebbles over, which do not make a complete set of ten.

Next. The sets of ten pebbles can be arranged in larger sets of ten times ten pebbles in each set, and the sets of ten pebbles over reserved which do not make a complete set of ten times ten pebbles.

Thirdly. The sets of ten times ten pebbles can be arranged in still larger sets of ten times ten times ten pebbles, and the sets over, of ten times ten pebbles reserved which do not make a complete set of ten times ten times ten pebbles, and so proceeding until the largest sets have been formed.

Thus, the whole heap of pebbles can be separated into different sets, and the groups in each set not exceeding nine in number. The first set consisting of single pebbles, not exceeding nine. The second, groups of ten pebbles in each set. The third, groups of ten times ten pebbles in each set. The fourth, groups of ten times ten times ten pebbles in each set, and so on, and ending with groups of the largest sets that can be formed.

Having shown how the number of a multitude of objects can be known by simply dividing them into different sets or classes, the second step is to assign names to the different sets, and thirdly to devise abbreviated characters or figures for denoting them. The various characters adopted by the most ancient nations for recording numbers were never employed as instruments of calculation. This double purpose was effected by that perfect system of numerical notation, by means of ten characters "with device of place," an invention for which the world is indebted to the intelligence of the ancient people of Hindustan. This method of naming and denoting numbers is called the denary system, as it reckons numbers by tens, tens of tens, and so on.

2. Numeration is defined to mean the method of expressing numbers by words, and notation is the art of denoting numbers by characters or figures.

Numbers are considered either as *concrete* or *abstract*. A *concrete number* is one which is employed to express any kind or species of things, and an *abstract number* is one which is considered apart from any species of things whatever. In considering numbers as abstract, an unit, or one, is the least whole number, and may be considered the primary element of numbers, and all abstract numbers can be formed by the successive additions of unity, beginning with unity.

The first number is named one; one and one is named two; two and one, three; three and one, four; four and one, five; five and one, six; six and one, seven; seven and one, eight; eight and one, nine.

These nine numbers are respectively represented by the characters 1, 2, 3, 4, 5, 6, 7, 8, 9.

Each of these nine figures, besides its intrinsic value, is assumed to have a local value depending on the place it occupies, reckoned from the figure in the first place on the right. The principle of the local value is simply this:—One figure being placed on the left of another, the place of the former is ten times as great as the place of the latter. Hence it follows that the first figure on the right being the place of units, a figure in the second place on the left of it is ten times its value in the first place, in the third place it is ten times that in the second place, in the fourth place it is ten times that in the third place, in the fifth place it is ten times that in the fourth place, and similarly for the succeeding places; so that a figure in the first, second, third, fourth, fifth, &c., places, denotes respectively units, tens of units, hundreds of units, thousands of units, tens of thousands of units, &c. They are designated numbers of the first, second, third, fourth, fifth, &c., orders respectively. Besides the nine significant characters, there is a tenth character, 0, named cipher, nought, or zero, which by itself has no numerical value, and denotes the absence of number in any place it may occupy in the notation of numbers. By means of these nine characters and the cipher, any number, however large, can be distributed into sets of units, tens, hundreds, &c., each set not exceeding nine, and can be exactly denoted by the assumed characters in their proper places.

The next number to nine, is nine and one (for which no single character is assumed), named ten, and is denoted by 10. And two tens, three tens, four tens, five tens, six tens, seven tens, eight tens, nine tens, are respectively denoted by 20, 30, 40, 50, 60, 70, 80, 90, and named twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety.

The natural numbers after 10, which increase by units, namely, ten and one, ten and two, ten and three, ten and four, ten and five, ten and six, ten and seven, ten and eight, ten and nine, are denoted by replacing 0 in 10 by 1, 2, 3, 4, 5, 6, 7, 8, 9, successively, and they become 11, 12, 13, 14, 15, 16, 17, 18, 19, respectively, and are named eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen.

And in the same manner the rest of the numbers of the second order are denoted, as 21, 22, 23, 24, 25, 26, 27, 28, 29; and so until the last, ninety and nine, or 99.

The next number, ninety and ten, is named one hundred, and is denoted by 100; and two hundreds, three hundreds, four hundreds, five hundreds, six hundreds, seven hundreds, eight hundreds, nine hundreds, are respectively denoted by 200, 300, 400, 500, 600, 700, 800, 900.

The number nine hundreds and one hundred is named one thousand, and is denoted by 1000. And in a similar manner two, three, four, five, six, seven, eight, nine thousands, respectively, are denoted by 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000.

And proceeding in the same manner, the successive numbers of the higher orders in the scale of notation are named and denoted. Thus:—

The number Ten thousands is denoted by 10,000.
One hundred thousands by 100,000.
One million by 1,000,000.
Ten millions by 10,000,000.
One hundred millions by 100,000,000.
One thousand millions by 1,000,000,000.
Ten thousand millions by 10,000,000,000.
One hundred thousand millions by 100,000,000,000.[1]

It is seldom in ordinary calculations that numbers are required beyond hundreds of thousands of millions.

The principle of the denary notation is not limited, and may be extended for the expression of numbers of any order. It is therefore obvious that the denary method of notation may be continued to any extent, and will suffice for the representation of any number, however great.

3. PROB. *To denote by figures any given number expressed in words; and conversely, to express in words any given number denoted by figures.*

First. Any number expressed in words can be denoted in figures by writing the figure which denotes the units in the first place, and the figures which denote the tens, hundreds, thousands, &c., respectively, in the second, third, fourth, &c., places on the left of the first place, with ciphers in those places where the figure of any order is wanting.

Conversely. A number denoted by figures can be expressed in words by first dividing the given number into periods of three or of six figures each, and writing the values of the figures which are in the first, second, third, fourth, &c., places as the units, tens, hundreds, thousands, &c., of which the number is composed.

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[1] The next unit in order is ten hundred thousand millions, or one thousand thousand millions, or one million of millions. Instead of considering the orders of higher numbers as successively formed by making ten units of one order equivalent to one unit of the next superior order, it has been found convenient to consider large numbers in periods of six figures each. The first period beginning with unity and extended to six figures is considered the period of units, the second the period of millions, the third the period of millions of millions, and so on, so that the units of the first, second, third, &c., periods in order will be as follows:

One, the primary unit denoted by 1.
One million denoted by 1 with 6 ciphers.
One million of millions, or a billion, is denoted by 1 with 12 ciphers.
One million of billions, or a trillion, by 1 with 18 ciphers.
One million of trillions, or a quadrillion, by 1 with 24 ciphers.
One million of quadrillions, or a quintillion, by 1 with 30 ciphers.
One million of quintillions, or a sextillion, by 1 with 36 ciphers.
One million of sextillions, or a septillion, by 1 with 42 ciphers.
One million of septillions, or an octillion, by 1 with 48 ciphers.
One million of octillions, or a nonillion, by 1 with 54 ciphers.
One million of nonillions, or a decillion, by 1 with 60 ciphers; and so on.

EXERCISES.

I.

Express in figures the following numbers:

1. Fifty-six; and sixty-five.
2. Three hundred and four; and five hundred and ninety-six.
3. Seven thousand seven hundred and seven; and nine thousand two hundred and seventy-three.
4. Twenty-seven thousand four hundred and fifty.
5. One hundred and ninety-eight thousand five hundred and sixty-seven.
6. Five millions five hundred thousand five hundred and five.
7. Ninety-five millions and fifty-nine thousand.
8. Seven hundred and ten millions seven hundred and ten thousand seven hundred and ten.
9. One thousand millions and one hundred thousand.

II.

Write in words at length the following numbers denoted by figures:

1. 57, 75, 80, 45, 21, and 79.
2. 101, 300, 425, 602, 620, and 999.
3. 1001, 1100, 1010, 4783, 2909, and 1800.
4. 21000, 20100, 48765, 18009, 18090, and 90009.
5. 123456, 700009, 709000, 100100, and 505050.
6. 7654321, 1002003, 5081000, and 9080706.
7. 29358764, 50800700, and 80053007.
8. 503100950 and 20000100010010.
9. 1000000200003001 and 5786732195946470.

III.

1. Express in figures the first hundred numbers with their respective names.
2. Write down the different ways in which each of the nine digits can be made up of two less numbers.
3. The number 27 is composed of 16 and 11. Write down all the other two numbers which can make up the number 27.
4. Write the smallest and the largest numbers possible with the first five characters, 0, 1, 2, 3, 4, in the denary notation, and express them in words.
5. How many tens, how many hundreds, how many thousands, and how many ten thousands are there in a million units?

6. Find all the integral numbers which can be formed out of the digits 0, 1, 2, 3, when one, two, three, four digits are taken respectively to form each number.

IV.

1. What methods did the ancient Greeks and Romans adopt to represent numbers? Were the Roman characters ever employed as instruments of calculation, as the figures in the decimal scale of notation?

2. From whence was the ordinary system of arithmetical notation by nine figures and zero derived? About what period was it introduced into Europe? What system did it supersede, and when did it begin to be generally used?

3. Define unit and number, and distinguish between abstract and concrete numbers.

4. Explain clearly the difference between the intrinsic values of each of the nine digits, and their local values in the denary scale; and shew that, with the aid of the tenth character zero, these digits are competent to express all numbers whatever, however great or however small.

5. In the denary scale of numeration, how many different words are employed in counting a million?

6. Give the derivation and explanation of the terms calculating, counting, cipher, algorithm.

7. In the denary system of notation explain why the number of digits cannot be more or less than the local value 10.

8. Specify clearly the advantages of the Indian method of arithmetic, (1) with respect to the powers of its notation, (2) of its operations.

9. State the considerations which render the sciences of number and space useful instruments for the training of the intellectual powers.

ADDITION AND SUBTRACTION.

ART. 1 The simplest elementary operations which follow the notation and numeration of numbers are named Addition and Subtraction.

By Addition is meant the process or operation by which two or more numbers are combined into one number, called their sum.

Subtraction is the reverse operation, by which a less number is taken from a greater, and the remainder, or difference, is found.

It is obvious that the operations of Addition and Subtraction are related, so that if 4 be added to 3, the sum is 7; and if 3 be subtracted from 7, the remainder is 4. Also, if 7 be taken from 7, the remainder is 0; and any number greater than 7, as 9, cannot be subtracted from 7.

It may also be remarked that Addition and Subtraction are the two fundamental operations in arithmetic, into which all others may be resolved; for whatever operation is to be performed, the change made in any given number must either be an increase or diminution of it.

The nature of the processes of numerical addition and subtraction imply that the abstract units employed are always the same.

2. PROB. *To find the sum of any two numbers consisting of single figures.*

This process will be effected by simply taking each unit of one of the numbers and successively adding it to the other. Thus the sum of the numbers 3 and 5 may be found by considering 3 as composed of three units, 1, 1, 1, and adding each in succession to the number 5.

The sum of 5 and 1 is 6,
,, 6 and 1 is 7,
,, 7 and 1 is 8,

that is, the sum of 3 and 5 is 8.

In a similar way the sum of 5 and 3 may be shewn to be 8, by considering 5 as composed of five units, 1, 1, 1, 1, 1, and adding each in succession to the number 3.

Hence the sum of 5 and 3 is the same as the sum of 3 and 5; or the sum of two numbers is the same in whatever way they may be added together.

The sums of every two numbers consisting of single figures may be found in this manner, and the results exhibited as follows:

The sum of 1 and 1 is 2	The sum of 2 and 1 is 3	The sum of 3 and 1 is 4
,, 2 ,, 3	,, 2 ,, 4	,, 2 ,, 5
,, 3 ,, 4	,, 3 ,, 5	,, 3 ,, 6
,, 4 ,, 5	,, 4 ,, 6	,, 4 ,, 7
,, 5 ,, 6	,, 5 ,, 7	,, 5 ,, 8
,, 6 ,, 7	,, 6 ,, 8	,, 6 ,, 9
,, 7 ,, 8	,, 7 ,, 9	,, 7 ,, 10
,, 8 ,, 9	,, 8 ,, 10	,, 8 ,, 11
,, 9 ,, 10	,, 9 ,, 11	,, 9 ,, 12

The sum of 4 and 1 is 5
,, 2 ,, 6
,, 3 ,, 7
,, 4 ,, 8
,, 5 ,, 9
,, 6 ,, 10
,, 7 ,, 11
,, 8 ,, 12
,, 9 ,, 13

The sum of 5 and 1 is 6
,, 2 ,, 7
,, 3 ,, 8
,, 4 ,, 9
,, 5 ,, 10
,, 6 ,, 11
,, 7 ,, 12
,, 8 ,, 13
,, 9 ,, 14

The sum of 6 and 1 is 7
,, 2 ,, 8
,, 3 ,, 9
,, 4 ,, 10
,, 5 ,, 11
,, 6 ,, 12
,, 7 ,, 13
,, 8 ,, 14
,, 9 ,, 15

The sum of 7 and 1 is 8
,, 2 ,, 9
,, 3 ,, 10
,, 4 ,, 11
,, 5 ,, 12
,, 6 ,, 13
,, 7 ,, 14
,, 8 ,, 15
,, 9 ,, 16

The sum of 8 and 1 is 9
,, 2 ,, 10
,, 3 ,, 11
,, 4 ,, 12
,, 5 ,, 13
,, 6 ,, 14
,, 7 ,, 15
,, 8 ,, 16
,, 9 ,, 17

The sum of 9 and 1 is 10
,, 2 ,, 11
,, 3 ,, 12
,, 4 ,, 13
,, 5 ,, 14
,, 6 ,, 15
,, 7 ,, 16
,, 8 ,, 17
,, 9 ,, 18

3. PROB. *To find the sum of two numbers consisting of more than one figure.*

AX. The sum of two or more numbers is equal to the sums of their respective parts taken together.

By the assumed notation every number is composed of units of the first, second, or third, &c., orders, named units, tens, hundreds, &c. And the sum of two or more numbers will be known by finding the sums of the units, the tens, the hundreds, and so on, and the aggregate of these sums will be the sum of the given numbers.

Let it be required to find the sum of two numbers, 358 and 287.

The number 358 consists of 3 hundreds, 5 tens, and 8 units,
and 287 consists of 2 hundreds, 8 tens, and 7 units.

Then, first, the sum of 7 units and 8 units is 15 units; or 1 ten and 5 units: reserve the 5 units.

Secondly. The sum of 1 ten, 8 tens, and 5 tens, is 14 tens; or 10 tens and 4 tens; or 1 hundred, and 4 tens: reserve the 4 tens.

Thirdly. The sum of 1 hundred, 2 hundreds, and 3 hundreds, is 6 hundreds.

Hence, the whole sum is 6 hundreds, 4 tens, and 5 units, or 645.

The process may be exhibited briefly, thus:—

$$\begin{array}{r}358\\287\\\hline 645\end{array}\ \text{sum.}$$

The sum of more numbers than two can be found in the same manner as the sum of two numbers.[1]

---

[1] When the columns of numbers to be added are large, the process of addition may be more easily effected by a method analogous to addition by counters. It relieves the memory from recollecting any number greater than 19; at the same time there is

And as ten units of any order are equal to one unit of the next superior order, it follows that the sum of two or more numbers of any order is found by the same operation as if they were of the first order, and all additions are reduced to the simple process of finding the sum of two or more single figures.

As an error is always possible in the performance of the process of addition, not much confidence can be placed in the so-called proofs of the process of addition. In the addition of numbers, if the sum be obtained by commencing with the last line of figures and adding upwards, in case any error be made, it may probably be detected by repeating the process, beginning with the first line of figures and adding downwards; and if the two results are the same, it is probable the addition is correct.

Besides the characters or symbols of numbers there are other assumed symbols which denote arithmetical operations.

The symbol + placed between two numbers indicates that the latter number is to be added to the former. Thus $3 + 5$ means that 5 is to be added to 3, and is read 3 plus 5.

The symbol = is used to indicate the equality or equivalence of two numbers; and is read "is equal to." Thus $3 + 5 = 8$, means that 5 added to 3 is equal to 8.

By the help of these two symbols of operation, any number can be expressed as the sum of the several orders of numbers of which it is composed; as, for example:—Thirty-five thousands, seven hundreds, forty and nine, can be thus expressed—

$$35749 = 30000 + 5000 + 700 + 40 + 9.$$

4. Prob.—*To find the difference of two numbers, each consisting of one or two figures.*

The difference of two small numbers will be found by subtracting from the greater number in succession as many units as are contained in the less.

Let it be required to find the difference between the two numbers 8 and 3.

3 is composed of three units, 1, 1, 1.

---

less trouble in practice, also the probability of error is considerably lessened by the use of small numbers only.

Instead of adding all the figures in each column at once, the sum of each column may be found as follows:

Add together the first two or more figures which produce a number between 10 and 19; place a mark for the 10, and add the units over to the next figures of the column, and mark the 10 as before, and add the units over to the next figures, and so on, marking all the tens until all the figures of the column have been added. Write the units over after the last addition under the column of units, and add to the second column of tens as many units as are equal to the number of marked tens in the first column. Proceed in the same way with the third and other columns.

Then subtracting each unit successively from 8,
1 taken from 8 leaves 7,
1 taken from 7 leaves 6,
1 taken from 6 leaves 5,
that is, 3 taken from 8 leaves a remainder 5.

And in a similar way the differences consisting of single figures, of two numbers, one of which does not exceed 10 nor the other 19, may be found, and the results exhibited as follows:

1 from 2 leaves 1
,, 3 ,, 2
,, 4 ,, 3
,, 5 ,, 4
,, 6 ,, 5
,, 7 ,, 6
,, 8 ,, 7
,, 9 ,, 8
,, 10 ,, 9

2 from 3 leaves 1
,, 4 ,, 2
,, 5 ,, 3
,, 6 ,, 4
,, 7 ,, 5
,, 8 ,, 6
,, 9 ,, 7
,, 10 ,, 8
,, 11 ,, 9

3 from 4 leaves 1
,, 5 ,, 2
,, 6 ,, 3
,, 7 ,, 4
,, 8 ,, 5
,, 9 ,, 6
,, 10 ,, 7
,, 11 ,, 8
,, 12 ,, 9

4 from 5 leaves 1
,, 6 ,, 2
,, 7 ,, 3
,, 8 ,, 4
,, 9 ,, 5
,, 10 ,, 6
,, 11 ,, 7
,, 12 ,, 8
,, 13 ,, 9

5 from 6 leaves 1
,, 7 ,, 2
,, 8 ,, 3
,, 9 ,, 4
,, 10 ,, 5
,, 11 ,, 6
,, 12 ,, 7
,, 13 ,, 8
,, 14 ,, 9

6 from 7 leaves 1
,, 8 ,, 2
,, 9 ,, 3
,, 10 ,, 4
,, 11 ,, 5
,, 12 ,, 6
,, 13 ,, 7
,, 14 ,, 8
,, 15 ,, 9

7 from 8 leaves 1
,, 9 ,, 2
,, 10 ,, 3
,, 11 ,, 4
,, 12 ,, 5
,, 13 ,, 6
,, 14 ,, 7
,, 15 ,, 8
,, 16 ,, 9

8 from 9 leaves 1
,, 10 ,, 2
,, 11 ,, 3
,, 12 ,, 4
,, 13 ,, 5
,, 14 ,, 6
,, 15 ,, 7
,, 16 ,, 8
,, 17 ,, 9

9 from 10 leaves 1
,, 11 ,, 2
,, 12 ,, 3
,, 13 ,, 4
,, 14 ,, 5
,, 15 ,, 6
,, 16 ,, 7
,, 17 ,, 8
,, 18 ,, 9

10 from 11 leaves 1
,, 12 ,, 2
,, 13 ,, 3

10 from 14 leaves 4
,, 15 ,, 5
,, 16 ,, 6

10 from 17 leaves 7
,, 18 ,, 8
,, 19 ,, 9

5. PROB.—*To find the difference of any two numbers consisting of more than two figures.*

The process depends on the two following axioms:

Ax. 1. The difference of two numbers is equal to the sum of the differences of their respective parts taken together.

Ax. 2. If two numbers be increased by equal numbers, their difference remains unaltered.

The given numbers being composed of figures of different orders, as units, tens, hundreds, &c.; if the units of every order of the less number be taken from the figures of the same order of the larger number, the sum of these partial differences will be the difference of the two given numbers.

But in cases where a figure of any order in the less number is greater than that of the same order in the larger number, the subtraction is not possible.

If this figure be increased by 10 units of the same order, the subtraction becomes possible; and if the figure of the next superior order in the less number be increased by 1, the larger and smaller numbers having been equally increased, the difference of the two numbers will be unaltered.

Let it be required to find the difference of 456 and 273.

Here 456 consists of 4 hundreds, 5 tens, and 6 units,
and 273 consists of 2 hundreds, 7 tens, and 3 units.

Now 3 units taken from 6 units leaves 3 units, the first partial remainder.

Next, 7 tens taken from 5 tens is impossible, but 10 tens added to 5 tens make 15 tens, and the subtraction becomes possible.

Then 7 tens taken from 15 tens leave 8 tens, the second partial remainder.

But as 10 tens are equal to 1 hundred, 1 hundred added to 2 hundreds make 3 hundreds, and 3 hundreds taken from 4 hundreds leave 1 hundred, the third partial remainder.

Hence the sum of these partial remainders will be the difference of the two numbers 456 and 273.

And the difference is 1 hundred, 8 tens, and 3 units, or 183.

The preceding process may be briefly exhibited, and the names of the orders of the figures may be omitted in performing the operation.

$$\begin{array}{l} 456 \\ 273 \\ \hline 183 \text{ difference.} \\ \hline \end{array}$$

The number 3 taken from 6 leaves 3. 7 cannot be taken from 5, add 10 to 5, which make 15; then 7 taken from 15 leaves 8. Next add 1 to 2, which makes 3, and 3 taken from 4 leaves 1; and the difference of the two numbers is 183.

The correctness of the process of subtraction may be readily verified; since of two numbers, the sum of their difference and the less number is equal to the greater.

6. The operation of the subtraction of one number from another is denoted by the sign —, called *minus*. Thus $8 - 5$ means that 5 is to be subtracted from 8, and is read 8 minus 5; also $8 - 5 = 3$ is read 8 minus 5 is equal to 3, or that the difference of 8 and 5 is 3.

When the sum or difference of two or more numbers is required to be considered as one number, the two numbers connected by the symbol of addition or subtraction are included in a parenthesis or a brace, thus $(5 + 2)$ and $(5 - 2)$, or $\{5 + 2\}$ and $\{5 - 2\}$, and sometimes by a line placed over them in this manner, $\overline{5 + 2}$ and $\overline{5 - 2}$.

One number may be considered the complement of another when the sum of the two numbers make up any given number.

7. Def.—The arithmetic complement of a number is defined to be the difference between any given number and the unit of the next superior order; as 6 is the arithmetic complement of 4, 47 of 53, 845 of 155, and so on, being the differences respectively of 4, 53, 155, and 10, 100, 1,000, the next superior units to these numbers. Conversely, also, 4, 53, 155 are the arithmetic complements of 6, 47, 845 respectively.[1]

The arithmetic complement of a number may always be found by subtracting the figure in the unit's place from 10 and the rest of the figures of the number from 9.

Since from the definition, the arithmetic complement of 155 is 845 $= 1000 - 155$, whence it follows that $155 + 845 = 1000$, or that the sum of any number and its arithmetic complement will always be equal to the unit of the next superior order.

Again, since $155 = 1000 - 845$; if 155 or its equivalent in terms of the arithmetic complement be subtractive, it may be written $\bar{1}845$, by placing the subtractive unit before the left digit of the arithmetic complement, as is done in the characteristics of logarithms, when they are subtractive.

8. Prob.—*If the arithmetic complement be added to any other number of the same number of figures, the sum will exceed the difference of the two numbers by an unit of the next superior order.*

If 155 be subtracted from 768, the remainder is 613, the difference between them.

But if the arithmetic complement 845 of 155, the less number, be added to 768, the greater, the sum will be 1613, one unit (1000 in this case) of the next superior order greater than the difference of the two numbers. By removing this unit, the number left will be equal to the difference of 768 and 155; so that the difference of two numbers can be found by addition. The formal removal may be avoided by writing the arithmetic complement thus, $\bar{1}845$, with the subtractive unit on the left, which when added to 768, the sum will be 613, the additive and subtractive units being equal to zero.[1]

---

[1] The arithmetic complement can be employed to find the difference of two numbers, as also the aggregate of several numbers when some of them are additive and some subtractive. It is employed with the greatest advantage in the arithmetic of logarithms, in cases where some logarithms are to be added and some to be subtracted in the same computation. Instead of finding the two sums and subtracting one from the other, the sum of the logarithms to be added and the arithmetic complements of the logarithms to be subtracted will give the correct difference of the sums of the additive and subtractive logarithms. The mode of writing the arithmetic complement is so simple that the arithmetic complement of a logarithm can be as readily written as the logarithm itself can be copied from the tables.

9. As numbers may be equal or unequal to one another, one number, or an aggregate of numbers, is said to be equal to, or equivalent to another number, when both are composed of the same number of units. And one number is said to be greater than another when there are more units in the former than in the latter; and the latter is said to be less than the former. The following axioms will be found to be useful in arithmetical reasonings.

1. Numbers are equal to one another which consist of the same number of units.

2. A whole number or integer is equal to the sum of all the parts of it.

3. Any whole number is greater than any part of it; and any part of a number is less than the number itself.

4. Numbers or aggregates of numbers which are equal to the same number are equal to one another.

5. If equal numbers be added to equal numbers, the sums are equal.

6. If equal numbers be added to unequal numbers: or if unequal numbers be added to equal numbers, the sums are unequal.

7. If equal numbers be taken from, or subtracted from equal numbers, the remainders are equal.

8. If equal numbers be subtracted from unequal numbers, or if unequal numbers be subtracted from equal numbers, the remainders are unequal.

9. If the same number be added to, and subtracted from another, the value of the latter is unaltered.[2]

---

[1] Ex. 1. Find the difference between 328011 and 23412 by means of the arithmetic complement.

328011
A. C. of 23412 is $\bar{1}76588$
By addition 304599 is the difference of the two numbers.

Ex. 2. Find the difference between the sum of 7970597, 1038037, and 3406424, and the sum of 3279120 and 8441042 by the arithmetic complement.

7970597
1038037
3406424
A. C. of 3272190 is $\bar{1}6727810$
A. C. of 8441042 is $\bar{1}1558958$

1701826 the difference required.

[2] As for example, $12 = 12 + 5 - 5 = 17 - 5 = 7 + 5$.

EXERCISES.

I.

Find the sums of the following sets of numbers, and express the results in words:

1. 5, 12, 20, 15, 35, and 50.
2. 37, 8, 99, 105, 250, 700, and 2.
3. 5000, 500, 20, 90, 9000, and 1.
4. 3584, 2796, 6416, 7204, 1897, and 8103.
5. 10000, 50505, 171717, 1008, and 201.
6. 235762, 105501, 287788, and 799999.
7. 1234567, 8765433, 6894703, 3105297, 5712843, and 4187157.
8. 57, 1876423, 1587, 999, and 3334447.
9. 100, 314500, 1237, 94, and 3000000.
10. 5876432100, 1284, and 387142659.

II.

1. Explain why in the addition of numbers the operation is begun at the unit's place. Is this necessary?

2. Shew that 5 and 7 make 12; and state explicitly every definition and axiom in the proof.

3. Add 3041 to 7198; and explain the operation.

4. Arrange the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, in three lines with three digits in each line, so that the sums of three digits taken in every possible direction may be always equal to 15.

5. Find the sum of all the numbers which can be formed with the figures 1, 2, 3, 4; all the four figures being taken to form each number.

III.

Find the differences of the following numbers, and verify the accuracy of the results:

1. 35 and 53.
2. 20000 and 999.
3. 478329 and 923874.
4. 1008425 and 100842.
5. 2784397 and 1234567.
6. 200000000 and 123456789.
7. 732584963 and 478342987.
8. 105723678 and 989456.
9. 10000001000 and 7077070077.

IV.

1. Subtract thirty millions twenty-six thousand and three, from forty-five millions seven thousand and twenty-one; and find what number must be added to the difference to make one hundred millions.

2. Add together four million twenty thousand and seventy-nine, twelve million two thousand and seven, and one million and five thousand, and subtract 16538107 from the sum.

3. What is the difference between the aggregate of 1050, 325, 1769, 150801, and a million? Shew that the same difference is obtained by taking one of the numbers from a million, another from the remainder, and so on for the rest of the numbers.

4. Subtract four billions thirty millions seventy-five thousand and eleven from 8876521856201, and write the difference in words.

5. Find the remainder after subtracting the numbers 44444, 9999, 666, 77, 1, in succession from 1000000.

6. Find the difference between the sum of 31845, 814, 10345, and the sum of 10014, 569, 1845.

7. Subtract the sum of 48002, 6100, and 5018462, from the difference of 1000000000 and 1234567890.

8. Subtract the difference between sixty billions and sixty thousands from the sum of fifty trillions and fifty millions.

V.

1. How may the accuracy of the process of subtraction be verified? Give an example.

2. Subtract 819 from 918, explaining the process.

3. Shew that the difference of 254 and 125 is the same as the difference when these numbers are each increased by 123.

4. What number subtracted from 670194 will leave 3825?

5. Half the sum of any two numbers is equal to the greater, and half the difference is equal to the less number. Apply this to find the two numbers whose sum is 521 and difference 175.

6. By how much does the sum of the numbers 27182818284 and 31415926535 exceed their difference?

VI.

1. Find the excess of 8765427 above 7634289 by means of the arithmetic complement; and explain the principle.

2. Find the result by means of the arithmetic complement of the numbers 578421, 854325, 104231, which are to be added, and of 325410, 684531, 213418, which are to be subtracted.

3. Find the difference of the sums of 256, 1875, 34210, and of 4008, 214, 56, by means of the arithmetic complement.

4. Find the difference of the aggregate of the numbers 1234567, 8543021, 5432147, and of 3245785, 5180072, by means of the arithmetic complement.

5. Write five numbers of seven places of figures each under each other, and beneath these write the arithmetic complements of four of them. Shew that the sum of the nine lines of figures will consist of the fifth line with 4 in the eighth place reckoned from the right-hand figure.

MULTIPLICATION AND DIVISION.

Art. 1. The multiplication of one number by another is the process of finding the number produced by the addition of one of them as many times as there are units in the other, as 5 multiplied by 3 or 3 times 5 is 15; and is the same as 5 added 3 times.

The number produced is called the product, the number to be multiplied the multiplicand, and the number indicating the number of times the multiplier. These are also called the factors of the product.

Division is the process of finding how many times a less number is contained in a greater, and is equal to the number of times the less number can be subtracted from the greater, as the number 5 is contained 3 times in 15, or 5 can be subtracted exactly three times from 15.

The greater number is named the dividend, the less number the divisor, and the number of times the divisor is contained in the dividend, the quotient.

The operations of multiplication and division are, one the reverse of the other; for since the product of 5 multiplied by 3 is 15, it follows that 15 divided by 5 gives 3 for the quotient.

The divisor and quotient are convertible, as for instance, if 15 be the dividend, and 3 the divisor, the quotient is 5; but if 5 be the divisor, the quotient is 3.

The names of dividend, divisor, and quotient in the operation of division correspond to those of product, multiplier, and multiplicand in the process of multiplication.

The processes of multiplication and division being one the reverse of the other, as the multiplicand multiplied by the multiplier produces the product, so the quotient multiplied by the divisor will reproduce the dividend when there is no remainder after the division. If, however, there be a remainder, it must be added to the product of the divisor and quotient to reproduce the complete dividend.

Any product is said to be a multiple of each factor, and each factor is called a submultiple of the product. Thus 15 is a multiple of each of the factors 5 and 3; and 5 and 3 are each submultiples of 15.

2. Prob. *To find the product of any two single figures.*

Let it be required to find the product of 5 and 3. The product of 5 multiplied by 3 means that 5 is to be repeated three times; that is, the sum of $5+5+5$ is 15. All the products of two single figures can be thus found by the simple process of addition.

The product of any two numbers is the same, whichever may be made the multiplier.

For instance, 5 multiplied by 3 gives the same product as 3 multiplied by 5.

The truth of this is evident from the fact, that if three rows of 5 counters be disposed in order under each other; in whichever way they may be counted, their sum will be same number of counters.

By counting them in one way, they constitute 3 rows of 5 counters in each row, or three times 5 counters.

By counting them the other way, they make 5 rows of 3 counters in each row, or 5 times 3 counters. Hence 3 times 5 counters is the same as 5 times 3 counters; or 3 times 5 is equal to 5 times 3.

This property is applicable for any two integral abstract numbers.

The following numbers exhibit the products of every two numbers not exceeding 9 :—[1]

[1] 1 time 1 is 1	2 times 1 is 2	3 times 1 is 3
,, 2 ,, 2	,, 2 ,, 4	,, 2 ,, 6
,, 3 ,, 3	,, 3 ,, 6	,, 3 ,, 9
,, 4 ,, 4	,, 4 ,, 8	,, 4 ,, 12
,, 5 ,, 5	,, 5 ,, 10	,, 5 ,, 15
,, 6 ,, 6	,, 6 ,, 12	,, 6 ,, 18
,, 7 ,, 7	,, 7 ,, 14	,, 7 ,, 21
,, 8 ,, 8	,, 8 ,, 16	,, 8 ,, 24
,, 9 ,, 9	,, 9 ,, 18	,, 9 ,, 27
4 times 1 is 4	5 times 1 is 5	6 times 1 is 6
,, 2 ,, 8	,, 2 ,, 10	,, 2 ,, 12
,, 3 ,, 12	,, 3 ,, 15	,, 3 ,, 18
,, 4 ,, 16	,, 4 ,, 20	,, 4 ,, 24
,, 5 ,, 20	,, 5 ,, 25	,, 5 ,, 30
,, 6 ,, 24	,, 6 ,, 30	,, 6 ,, 36
,, 7 ,, 28	,, 7 ,, 35	,, 7 ,, 42
,, 8 ,, 32	,, 8 ,, 40	,, 8 ,, 48
,, 9 ,, 36	,, 9 ,, 45	,, 9 ,, 54
7 times 1 is 7	8 times 1 is 8	9 times 1 is 9
,, 2 ,, 14	,, 2 ,, 16	,, 2 ,, 18
,, 3 ,, 21	,, 3 ,, 24	,, 3 ,, 27
,, 4 ,, 28	,, 4 ,, 32	,, 4 ,, 36
,, 5 ,, 35	,, 5 ,, 40	,, 5 ,, 45
,, 6 ,, 42	,, 6 ,, 48	,, 6 ,, 54
,, 7 ,, 49	,, 7 ,, 56	,, 7 ,, 63
,, 8 ,, 56	,, 8 ,, 64	,, 8 ,, 72
,, 9 ,, 63	,, 9 ,, 72	,, 9 ,, 81

3. Prob. *To find the product of two numbers, one of which consists of one figure only.*

The process depends on the following

Axiom. The product of any two numbers is equal to the sum of the products obtained by multiplying one of the numbers by each of the parts into which the other can be divided.

---

[1] If the products of two numbers were extended to 12 times 12, or to 20 times 20, and committed to memory, additional facilities would be acquired in the processes of multiplication and division of numbers.

Let it be required to find the product of 5342 by 4.

Here 5342 = 5000 + 300 + 40 + 2.

And 4 times 5000 = 20000.
4 times 400 = 1600.
4 times 40 = 160.
4 times 2 = 8.

The sum of the partial products is 21368, the product of 5342 by 4.

In practice, the process is thus exhibited, omitting the ciphers—

$$\begin{array}{r} 5342 \\ \underline{4} \\ \underline{21368} \end{array}$$

The addition and multiplication are performed at each step of the process, in this manner:

First. 4 times 2 is 8 ; write 8 under the place of units.

Secondly. 4 times 4 is 16 ; write 6 in the place of tens, and reserve the 1.

Thirdly. 4 times 3 is 12, and adding the 1 reserved make 13 ; write 3 in the third place, and reserve the 1.

Fourthly. 4 times 5 is 20, and adding the 1 reserved make 21, which write in the next place, and the product is 21368.

4. PROB. *To find the product of any two numbers.*[1]

---

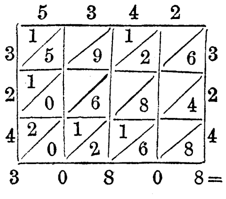
[1] The following method of multiplication is found in the commentary of Ganesa on the Lilavati, and was the method adopted by the Arabians, by whom it seems to have been preferred. It was also adopted by the Persians with some slight modification, and, lastly, by the Italians.

Form a parallelogram whose length and breadth respectively contain as many units as there are digits in the multiplicand and multiplier, and draw parallel lines through the points of division, thus dividing the figure into equal squares, and lastly, let diagonals be drawn in the same direction in these squares.

Write the digits of the multiplicand and multiplier along the length and breadth of the parallelogram, placing each digit opposite to a square beginning with the highest places from the same angle.

Multiply the digits of the multiplicand and multiplier, and place the units of each product in the lower, and the tens in the upper half of the square which is common to the two figures which are multiplied together.

Ex. Multiply 5342 by 324.



1 7 3 0 8 0 8=1730808, the product.

The operation could be performed from left to right instead of from right to left, as the products of every two figures of the multiplicand and multiplier would still occupy the same places.

Let it be required to find the product of 5342 and 324.

The multiplier 324 = 300 + 20 + 4.

If the multiplicand be first taken 4 times, then 20 times, and thirdly 300 times, the sum of these three partial products will be equal to 324 times 5342, or the product required.

The process may be thus exhibited:

$$\begin{array}{r l}
5342 & \\
324 & = 300 + 20 + 4. \\
\hline
21368 & = 4 \text{ times } 5342. \\
106840 & = 20 \quad ,, \\
1602600 & = 300 \quad ,, \\
\hline
1730808 & = 324 \quad ,, \\
\hline
\end{array}$$

In practice it is usual to omit the ciphers, taking care to place the first figure of each partial product under the figure which forms the multiplier for that product.[1]

The product of two numbers can also be found by multiplying one of them by the factors into which the other can be divided.

Thus the product of 5342 by 324 can be found by multiplying 5342 by 4, 9, 9 in succession, the factors of 324.

$$\begin{array}{r l}
5342 & \\
4 & \\
\hline
21368 & = 4 \text{ times } 5342. \\
9 & \\
\hline
192312 & = 36 \text{ times } 5342. \\
9 & \\
\hline
1730808 & = 324 \text{ times } 5342. \\
\hline
\end{array}$$

---

[1] The following is the proof of the process of multiplication "by casting out the nines." It depends on the property that "any number divided by nine will leave the same remainder as the sum of its digits divided by nine."

First. Cast out the nines from the sums of the digits of the multiplicand and of the multiplier, and reserve the remainders.

Secondly. Multiply these two remainders together and cast out the nines from their product, and reserve the remainder.

Thirdly. Cast out the nines from the sum of the digits of the product, and reserve the remainder.

Fourthly. Then if these two remainders be equal, the process of the multiplication is correct.

This so-called proof is defective as a proof in the following particulars, as it fails to detect errors in the product—

1. If the order of figures in the product be misplaced, as 37 for 73.
2. If errors be made which counterbalance each other, as 35 written for 62, the sums of the digits in each case being the same.

If 9 be written for 0, or 0 for 9, or either be omitted or inserted too often.

The proof of multiplication "by casting out the elevens" is not liable to so many chances of failure as the proof "by casting out the nines."

5. *A number is multiplied by* 10, 100, 1000, *&c., by annexing one, two, three, &c., ciphers to the right-hand figure of the multiplicand.*

It will be seen that such multiplications are in fact the direct consequence of the assumed notation. For when one, two, three, &c., ciphers are respectively annexed to any number, each of its digits is removed one, two, three, &c., places respectively towards the left, and in consequence of the assumed principle of the local value, each figure becomes 10, 100, 1000 times respectively as great as it was before.

It may also be noted, if both multiplicand and multiplier end in one or more ciphers, the product will be found by multiplying the significant figures of the two numbers, and annexing to the product as many ciphers as are equal to those on the right of the multiplier and multiplicand.

There are numerous devices whereby labour may in some instances be avoided in finding the product of two numbers, some of which are the following:

1. Any number is multiplied by 9, or $10 - 1$, by subtracting the given number from 10 times the number; and by 11, or $10 + 1$, by adding the given number to 10 times the number.

2. Any number can be multiplied by 99, 999, 9999, &c., by annexing 2, 3, 4, &c., ciphers to the multiplicand, and subtracting the multiplicand itself from this product.

And in a similar manner any number can be multiplied by another composed of a repetition of the figure 9 with any other figure in the highest place.[1]

6. The symbol × is assumed to indicate multiplication of numbers, as 3×5 denotes 3 multiplied by 5, and $3 \times 5 = 15$ means the product 3 and 5 is equal to 15. Also, $3 \times 5 + 2 = 17$ denotes that 2 added to the product of 3 and 5 is equal to 17, and $3 \times 5 - 2 = 13$, that 2 subtracted from the product of 3 and 5 is equal to 13. Also, when two or more equal factors are to be multiplied together, the product may be briefly expressed by writing at the upper part on the right of one of them a small figure, denoting the number of equal factors. Thus 3×3 is denoted by 3^2; $3 \times 3 \times 3$ by 3^3; and $3 \times 3 \times 3 \times 3$ by 3^4. Also $3^4 \times 5^6$ will express the product arising from 4 factors each equal

---

[1] Ex. Find the product of 34578 by 999.

Here $34578000 = 1000$ times 34578.
and $\underline{34578 = \quad 1}$,,
$\underline{34543422 =}$ 999 times 34578

Ex. Find the product of 34578 by 699.

Here $699 = 700 - 1$.
And $24204600 = 700$ times 34578
$\underline{34578 = \quad 1}$,,
$\underline{24170022} = 699$ times 34578.

to 3 and 6 factors each equal to 5. Sometimes $3^4 \times 5^6$ is more briefly denoted by $3^4.5^6$.

7. Prob. *To find the quotient of two numbers which consist of one or of two digits.*

Let it be required to find the quotient arising from 15 divided by 5. The quotient denotes the number of times the divisor is contained in, or can be subtracted from, the dividend.

Let 5 be subtracted successively from 15. First. 5 subtracted from 15 leaves 10. Secondly. 5 from 10 leaves 5. Thirdly. 5 from 5 leaves 0. So that 5 can be subtracted 3 times from 15, or 5 is contained 3 times in 15, or 15 divided by 5 gives 3 for the quotient.

This will also appear from the fact that division being the reverse of multiplication, since 3 times 5 is 15, it follows that 5 is contained 3 times in 15.

The following statements exhibit how many times each of the nine digits is exactly contained in numbers consisting of one or two digits:

1	divided by 1	gives	1	2	divided by 2	gives	1	3	divided by 3	gives	1
2	,,	,,	2	4	,,	,,	2	6	,,	,,	2
3	,,	,,	3	6	,,	,,	3	9	,,	,,	3
4	,,	,,	4	8	,,	,,	4	12	,,	,,	4
5	,,	,,	5	10	,,	,,	5	15	,,	,,	5
6	,,	,,	6	12	,,	,,	6	18	,,	,,	6
7	,,	,,	7	14	,,	,,	7	21	,,	,,	7
8	,,	,,	8	16	,,	,,	8	24	,,	,,	8
9	,,	,,	9	18	,,	,,	9	27	,,	,,	9

4	divided by 4	gives	1	5	divided by 5	gives	1	6	divided by 6	gives	1
8	,,	,,	2	10	,,	,,	2	12	,,	,,	2
12	,,	,,	3	15	,,	,,	3	18	,,	,,	3
16	,,	,,	4	20	,,	,,	4	24	,,	,,	4
20	,,	,,	5	25	,,	,,	5	30	,,	,,	5
24	,,	,,	6	30	,,	,,	6	36	,,	,,	6
28	,,	,,	7	35	,,	,,	7	42	,,	,,	7
32	,,	,,	8	40	,,	,,	8	48	,,	,,	8
36	,,	,,	9	45	,,	,,	9	54	,,	,,	9

7	divided by 7	gives	1	8	divided by 8	gives	1	9	divided by 9	gives	1
14	,,	,,	2	16	,,	,,	2	18	,,	,,	2
21	,,	,,	3	24	,,	,,	3	27	,,	,,	3
28	,,	,,	4	32	,,	,,	4	36	,,	,,	4
35	,,	,,	5	40	,,	,,	5	45	,,	,,	5
42	,,	,,	6	48	,,	,,	6	54	,,	,,	6
49	,,	,,	7	56	,,	,,	7	63	,,	,,	7
56	,,	,,	8	64	,,	,,	8	72	,,	,,	8
63	,,	,,	9	72	,,	,,	9	81	,,	,,	9

8. Prob. *To divide any number by a number of one digit.*

The process depends on the following

Axiom. One number is contained in another as many times as the former is contained in the several parts into which the latter can be divided.

Let it be required to divide 7583 by 5.

Since division is the reverse of multiplication, the process of division is performed by reversing the operation of multiplication, with this difference, that the process of division begins with the figure on the left hand of the dividend and proceeds towards the right, but the process of multiplication begins with the figure on the right of the multiplicand and proceeds towards the left.

The dividend 7583 = 7000 + 500 + 80 + 3.

First. 7 thousands divided by 5 units give 1 thousand, the first partial quotient, and 2 thousands, or 20 hundreds, over.

Secondly. 20 hundreds added to 5 hundreds make 25 hundreds, and 25 hundreds divided by 5 units give 5 hundreds, the second partial quotient.

Thirdly. 8 tens divided by 5 units give 1 ten the third partial quotient, and 3 tens, or 30 units, over.

Fourthly. 30 units added to 3 units make 33 units, and 33 units divided by 5 units give the fourth partial quotient 6 units, and 3 units remainder. The process is thus exhibited :—

$$\begin{array}{r} 5\overline{)7583} \\ \overline{1516} \end{array} \text{ — 3 remainder.}$$

In practice the names of the orders of units are omitted, and the process is performed naming only the figures. 7 divided by 5 gives quotient 1 and 2 over. Secondly, 25 by 5 gives 5 and no remainder. Thirdly, 8 by 5 gives 1 and 3 over. Fourthly, 33 by 5 gives 6, and 3 the last remainder.

9. Prob. *To divide any greater number by a less number.*

Let it be required to divide 1730808 by 5342.

Since it has been shewn (Art. 4) that the sum of 300 times, 20 times, and 4 times 5342 is equal to 1730808, it follows that 300 times, 20 times, and 4 times 5342 successively subtracted from 1730808 leaves the remainder 0.

And the method whereby 300, 20, and 4 can be found from 1730808 and 5342 will be simply a reversal of the process by which 1730808 was found from 5342 and 324.

First. The divisor 5342 is contained 300 times in the dividend 1730808.[1] When 300 times 5342 is subtracted from the dividend, the remainder is 128238.

---

[1] Some difficulty may be experienced in the first attempts to find the exact number of times the divisor is contained in the successive partial dividends. If it be borne in mind that the product of the divisor by any quotient figure, when subtracted from any dividend, must always leave a remainder less than the divisor, the difficulty may soon disappear.

By using trial divisors (as in the case above, 5000 and 6000, one less and the other greater than the given divisor), the limits can be found within which the correct quotient figure lies. The best and most ready mode of acquiring facility in the process of division, is by finding the product of two numbers, and then reversing the operation.

Secondly. The divisor 5342 is contained 20 times in the second dividend 128208. When 20 times 5342 is subtracted from this dividend, the remainder is 21368.

Thirdly. The divisor 5342 is contained 4 times in the third dividend 21368. When 4 times 5342 is subtracted from this dividend, the remainder is 0.

The process may be thus exhibited:

$$\begin{array}{r@{}l} 5342)1730808 & (300+20+4 \\ \underline{1602600} & \\ 128208 & \\ \underline{106840} & \\ 21368 & \\ \underline{21368} & \\ 0 & \end{array}$$

In practice the value of the units and the ciphers are omitted, and the process is thus performed:—

First. The divisor 5342 is found to be contained 3 times in the first five figures of the dividend. The product of 5342 by 3 is subtracted, and to the remainder 1282 is annexed 0, the sixth figure of the dividend.

Secondly. The divisor 5342 is contained 2 times in 12820, and the product of 2 times 5342 is subtracted from this number, and to the remainder 2136 is annexed 8, the seventh figure of the dividend.

Thirdly. The divisor 5342 is contained 4 times in 21368, and 4 times 5342 subtracted from 21368 leaves no remainder.

And 5342 is contained 324 times exactly in 1730808.

The process is thus shewn:

$$\begin{array}{r@{}l} 5342)1730808 & (324 \\ \underline{16026} & \\ 12820 & \\ \underline{10684} & \\ 21368 & \\ \underline{21368} & \\ 0 & \end{array}$$

When there are one or more ciphers at the right hand of the divisor they may be omitted, and as many figures from the right hand of the dividend, and the division performed with the remaining figures, and at the end of the process, to the remainder must be annexed the figures omitted in the dividend, to make up the whole remainder.

10. Any number is divided by 10, 100, 1000, &c., by marking off one, two, three figures from the right hand of the dividend. These will be the remainder, and the rest will form the quotient.

Any number can be divided by 9, 99, 999, 9999, &c., by successively dividing the given number by 10, by 100, by 1000, by 10,000, &c., respectively, and taking the sum of the successive quotients for the true quotient, and the sum of the successive remainders for the true remainder; except when the sum of the latter exceeds the next higher unit; in that case both the quotient and remainder must be increased by unity.[1]

Any number can be divided by 5, 25, 125, &c., by multiplying the dividend by 2, 4, 8, &c., and marking off towards the left from the unit's place, one, two, three, &c., figures. The figures on the left will be the quotient, and those on the right will be 2, 3, 4, &c., times the respective remainders.[2]

11. Prob. *To divide a number by the factors of the divisor, and to determine the correct remainder after the division.*

Divide 109958 by 5, 7, 11, the factors of 385, and determine the correct remainder.

5)109958
7)21991 — 3 first remainder.
11)3141 — 4 second remainder.
285 — 6 third remainder.

The quotient is 285, with remainders 3, 4, 6, after the divisions by 5, 7, 11 respectively.

Every unit in the 2nd line is 5 times each unit in the 1st line.
,, 3rd ,, 7 ,, ,, 2nd line.
,, 4th ,, 11 ,, ,, 3rd line.

Hence—
3, the remainder from the 1st line, is 3 units of the 1st line.
4, ,, 2nd ,, 5 times 4 or 20 ,,
6, ,, 3rd ,, 7 times 5 times 6 or 210 ,,

The sum of these partial remainders is 233, the whole remainder.

The process may be verified by dividing 109958 by 385.

The product of two numbers is the same, whichever of the two be made the multiplier, as also of any number of factors, in whatever order the numbers may be multiplied. It follows, that if a dividend be divided by several divisors in succession, the quotient and the remainder are the same, in whatever order the separate divisions are performed.

---

[1] Ex. Divide 65874 by 99.

100)658,74
6,58
6
665,39

Here the sum of the partial remainders is 138, and both the quotient and remainder must be increased by unity. The truth of the process may be verified by performing the division of 65874 by 99 in the ordinary way.

[2] George Suffield, M.A., late Fellow of Clare College, published a small tract entitled, "Synthetic Division in Arithmetic, with some introductory remarks on the periods of circulating decimals." This method is useful for obtaining quotients when a very large number of figures is required. It contains some brief and ingenious methods for converting ordinary fractions into repeating decimals.—*Macmillan and Co., Cambridge and London*, 1863.

12. The symbol $\div$ is assumed to denote the division of one number by another; thus $18 \div 3$ means 18 divided by 3, and $18 \div 3 = 6$ is read 18 divided by 3 is equal to 6.

A more convenient notation has been assumed to denote the division of one number by another, by simply placing the dividend above the divisor with a line of separation between them. Thus $\frac{18}{3} = 6$ denotes that 18 divided by 3 is equal to 6. By means of this notation for representing the division of one number by another, the complete quotients in all cases of division may be indicated. Thus, $\frac{18}{3}$, and $\frac{19}{3}$, indicate the respective quotients of 18 divided by 3, as well as 19 divided by 3, whether the dividend be exactly divisible by the divisor or not.

13. The following axioms will be found useful in arithmetical reasonings:—

1. Equal numbers multiplied by equal numbers are equal.
2. Equal numbers multiplied by unequal numbers, or unequal numbers multiplied by equal numbers, are unequal.
3. Equal numbers divided by equal numbers are equal.
4. Equal numbers divided by unequal numbers, or unequal numbers divided by equal numbers, are unequal.
5. Any number multiplied and divided by the same number remains unaltered.
6. When a dividend and a divisor are both multiplied or both divided by the same number, the quotient is unaltered.[1]

ON OTHER SYSTEMS OF NUMERICAL CALCULATION.

14. In the preceding pages has been shewn the method of denoting numbers by means of ten characters with device of place, and of performing the processes of addition and subtraction, multiplication and division with them as instruments of calculation. The local value of this system of arithmetic being ten, is only one of the possible systems of notation.

Any other number than ten might have been assumed for the local value, and hence it follows that other systems of numerical notation can be devised analogous to the denary system, and the character 0 will be essential in every one of them, whatever may be the local value of the scale of notation.

In every system of notation the number of characters which have an intrinsic value as well as a local value cannot be greater nor less than the number which denotes the local value of every two consecutive characters. The local value of any system determines the number of independent characters to be employed in that system; and the notation of numbers will become confused when the number of characters is greater in number than the local value, and impossible when the number is less.

---

[1] In the Lilavati, directions are given "to abridge the dividend and divisor by equal numbers, whenever that is practicable," or, in other words, to divide the dividend and divisor by any number by which they are divisible exactly.

As in the denary system the number of characters must be ten, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, the number ten being represented by 10, so in the septenary system, in which the local value is 7, the number of characters must be seven, 0, 1, 2, 3, 4, 5, 6, the number seven being denoted by 10.

The following characters are required in the first eleven systems :—

Binary system,	local value	2,	requires	2	characters, 0, 1.
Ternary	,,	3,	,,	3	,, 0, 1, 2.
Quaternary	,,	4,	,,	4	,, 0, 1, 2, 3.
Quinary	,,	5,	,,	5	,, 0, 1, 2, 3, 4.
Senary	,,	6,	,,	6	,, 0, 1, 2, 3, 4, 5.
Septenary	,,	7,	,,	7	,, 0, 1, 2, 3, 4, 5, 6.
Octenary	,,	8,	,,	8	,, 0, 1, 2, 3, 4, 5, 6, 7.
Nonary	,,	9,	,,	9	,, 0, 1, 2, 3, 4, 5, 6, 7, 8.
Denary	,,	10,	,,	10	,, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
Undenary	,,	11,	,,	11	,, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t.
Duodenary	,,	12,	,,	12	,, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e.

The letters t, e, being taken to denote the numbers ten and eleven, twelve being denoted by 10. And so for any other system of a larger local value.

The operations of addition, subtraction, multiplication, and division in any system may be performed in exactly the same manner as the same operations are performed in the denary system of arithmetic. A number expressed in one scale of notation can be transformed into any other scale in which the local value is different. It will, however, be found more easy in practice to transform the given number of any scale into another in the denary scale, and this number into that of the scale required.[1]

---

[1] To transform 3542 from the septenary to the denary scale.

Here $3542 = 3.7^3 + 5.7^2 + 4.7 + 2 = 1304$.

The operation may be exhibited thus :—

$$\begin{array}{r} 3542 \\ 7 \\ \hline 26 \\ 7 \\ \hline 186 \\ 7 \\ \hline 1304 \\ \hline \end{array}$$

And 3542 in the septenary = 1304 in the denary scale.

Conversely. To transform 1304 from the denary to the septenary scale.

Since $1304 = 3.7^3 + 5.7^2 + 4.7 + 2 = 3542$, Each different system of arithmetical notation admits of fractions analogous to decimal fractions as well as ordinary fractions. All other scales of notation except the denary, are for the most part subjects of curiosity rather than utility, yet not altogether useless from an intellectual point of view, as they supply instances of the application and extension of a general principle.

It may however be noted, that the duodenary scale is sometimes employed for finding, by means of the duodecimal division of the foot, the areas of surfaces and the volumes of solids.

It is obvious that the digits 3, 5, 4, 2, will be found by successively dividing 1304 by the local value 7.

```
7)1304
  ----
7)186 — 2
  -------
 7)26 — 4
   ------
  7)3 — 5
    -----
    0 — 3
```

And 1304 in the denary = 3542 in the septenary scale.

Ex. To find the sum of 2345 and 7164, and the difference of 3542 and 1164 in the septenary scale:—

```
 2345          |    3542
 1164          |    1164
 ----          |    ----
 3542 sum.     |    2345 difference.
 ----          |    ----
```

To find the product of 3542 and 1164; and the quotient of 4566111 divided by 1164 in the septenary scale:—

```
      3542        |  1164)4566111(3542 quotient.
      1164        |       3555
     -----        |       -----
     21131        |       10111
    31545         |        6246
    3542          |        -----
   3542           |        5321
   -------        |        5052
   4566111 product.|       -----
   -------        |        2361
                  |        2361
                  |       -----
```

EXERCISES.

I.

1. Upon what principle is the ordinary method of multiplication based? Is there any advantage in beginning with the right-hand figure (1) of the multiplicand, (2) of the multiplier?

2. Explain the method for the multiplication of two numbers, each consisting of several figures, and multiply 30071 by 20590, explaining the reason for each part of the process.

3. Shew that the product of any number by a composite number may be found by multiplying by its factors successively. Ex. Multiply 3605 by 21.

4. Shew that the product of 2356 by 135 may be found by multiplying 2356 by 11 and that product by 12, and adding 3 times the multiplicand to the product.

5. Exemplify the truth that two or more factors will give the same product in whatever order they are multiplied.

6. Any number may be multiplied by 5, 25, 125, &c., by annexing 1, 2, 3, &c., ciphers respectively to the number, and then dividing it by 2, 4, 8, &c. Explain the reason of this rule.

7. In the multiplication of numbers, how do you prove the correctness of the operation by casting out the nines? Explain and give reasons for the rule, and shew the errors to which it is liable.

II.

Find the products of the following numbers:

1. 2567 and 256 : 57834 and 123456 : 18007 and 70081.
2. 2187642 and 5874568 : 9876423 and 99999.
3. 2500 and 27 : 2589 and 1200 : 786300 and 4000.
4. 8426000 and 8426000 : 783410 and 70001.
5. 87, 97, 101, 113, and 6845.
6. 12, 123, 1234, 12345, 123456, and 1234567.

III.

Multiply the following sets of numbers, each by means of 3 lines of multiplication :—

1. 578657 by 729819.
2. 1234567890 by 4321089.
3. 328560 by 121711.
4. 43002073252 by 133112111.
5. 12345678 by 28814412.
6. 30040769503 by 172814412.

IV.

1. Shew that the product of two odd numbers is an odd number; the product of two even numbers an even number; and the product of an odd and an even number an even number.

2. How many times must 1874 be added to itself to make a total of 163038?

3. Find the sum of twenty consecutive numbers beginning with 18547 by the shortest process.

4. Shew how to multiply 32165 by 648 in two lines.

5. Multiply 967548 by 492 in two lines.

6. Determine a number by which 4389 can be multiplied, so that the product may lie between the limits of 500000 and 1000000.

7. Express MMDCXCIX. and CCCXXIX. in the ordinary numerical characters; find their product, and express the result in the Roman characters.

8. The product of 75 by 43 is 3225; how much must be added to it to obtain the product of 77 by 43?

V.

1. What is the object of division? In what cases may it be considered as a shortened subtraction?

2. What arithmetical operation bears the same relation to division that addition bears to multiplication?

3. The quotient of two numbers is the same as the quotient of any equimultiples or equisubmultiples of the given numbers. Are the remainders the same in the three cases? Give an example.

4. Explain clearly the process of dividing one large number by another. Ex. Divide 143255 by 4093.

5. A number can be divided by 5, 25, 125, &c., by multiplying the dividend by 2, 4, 8, &c., and marking off towards the left from the unit's place, one, two, three, &c., figures. The figures on the left will be the quotient, and those on the right will be 2, 4, 8, &c., times the respective remainders. Explain the reason of this.

6. Express by means of figures and symbols of operation—
    1. That the number eight increased by seven is equal to the product of five multiplied by three.
    2. That the number ten diminished by four is equal to the same number as thirty divided by five.

VI.

Find the quotient of the following numbers, and verify the truth of each process:—

1. 3745687 divided by each of the nine digits respectively.

2. 28796478 by the factors of each of the numbers 49, 64, 120, and 132 respectively.

3. 1628756 by 3840, by 38400, and by 384000 respectively.

4. 287460000 by 687000, and 28400 by 169.
5. 1000000 by 1001, and 40012805 by 287000.
6. 1809456 by the sum of 2845, 1070, and 3578.
7. 2100000 by the difference between 2500 and 5200.
8. 278459675 by 9, 99, 999, 9999, and 99999 respectively.

VII.

1. If the quotient be 5000 when the divisor is 2001 and the remainder 100, what is the dividend?

2. What number divided by 528 will give 36 for the quotient and leave 44 as a remainder?

3. If the dividend be 784622 and the quotient be 4044, what is the divisor and the remainder?

4. If the quotient be 194, the divisor 4044, and the remainder 87, what is the dividend?

5. If the divisor be half the quotient, and the quotient three times the remainder, what is the dividend when the remainder is 10000?

6. If the divisor be seven times, and the quotient ten times the remainder, what is the dividend when the remainder is 1234?

7. The quotient is equal to 6 times the divisor, and the divisor to 6 times the remainder, and the three together amount to 516; find the dividend.

VIII.

1. Divide the product of the nine digits by their sum.

2. What number multiplied by 64 produces one million?

3. Multiply 50738 by 4620, and verify the result by dividing the product by the factors of the multiplier.

4. The product of two numbers is 1270630, and half of one of them is 3129; what is the other number?

5. Add together the sum, difference, product, and quotient of the two numbers 825 and 9318375.

6. What are the two *nearest* numbers in excess and defect to 1376429, which can be divided by 36300 without remainder?

7. Multiply 438375 by 287298, and divide the result by 12375.

8. What number divided by 37 will give the same quotient as 816405 divided by 111.

9. What number is that which when increased 13 times itself amounts to 20005006?

IX.

1. The product of two numbers is 760069388, one of them being 26078; shew that their sum divided by 1534 is 36.

2. Find the least number which must be added to 78421 to make it an exact multiple of 245.

3. Shew that the sum of all the numbers which can be formed by any three figures, as 2, 5, 6, taken together, is divisible by the sum of the three figures 2, 5, and 6.

4. If the number 12 be increased by its double and be multiplied by its treble, what multiple of the original number is the product?

5. Explain how any number may be multiplied by 11 by addition, and divided by 9 by subtraction. Ex. 45789 × 11 and 45789 ÷ 9.

6. Find the sum of all the numbers between 5000 and 5025 which are not divisible by any one of the numbers 2, 3, 5, 7, or 11.

7. What equimultiples of 12, 18, 24, added together, make up 7776?

8. Divide the number 960 into four parts so that the parts may be equimultiples of 1, 3, 5, and 7.

9. Divide the number 1014 into three such parts such that 88 the excess of the first part above the second may be equal to the excess of the second above the third.

X.

1. If 19 be subtracted as often as possible from 1499, what will be the final remainder?

2. How many times must 236 be subtracted from 114309 that the final remainder may be 73245?

3. Divide 10149 by 7 and the quotient by 5; thence deduce the true remainder, and shew that it is the same as after the division of 10149 by 35.

4. If division by a composite number be performed by successively dividing by its prime or composite factors, shew how the complete remainder may be found. Ex. 1437281 divided by 105.

5. The product of two numbers is 373625, the greater of them is 875; find the sum and the difference of the two numbers.

6. Shew that the product of 3846 and 705 is equal to the quotient arising from 51517170 divided by 19.

7. What number multiplied by 345 will give the same product as 2415 multiplied by 197.

XI.

1. Any number composed of three consecutive significant figures is divisible by 3.

2. A number of six digits consisting of the repetition of any one of the nine digits is divisible by 7, 11, and 13.

3. All numbers expressed by any number of the same digits are respectively divisible by the sum of the digits. Exemplify this property by examples consisting of 5, 7, and 12 digits respectively.

4. If any numbers consisting of one, two, three, four, &c., figures be multiplied by 9, the sums of the digits of the successive products are either 9 or a multiple of 9.

5. In a number consisting of two digits, if the figure in the place of tens be double that in the place of units, the number is divisible by 7, and the quotient is the sum of the two figures. Explain why this is so.

6. If any given number be separated into periods of three figures

each (the last may consist of one or two figures) beginning from the unit's place, and the sum of those periods be divided by 37, the remainder will be the same as when the given number itself is divided by 37. Ex. Let 5623456 be divided by 37.

7. If any odd number not ending with 5 be made the multiplier, a product may be found which shall consist of the repetition of any one of the nine digits. Verify this property by taking 13 for the multiplier, when the products respectively consist of repetitions of each of the nine digits.

XII.

Transform the following numbers and reverse each operation:

1. 368 from the denary to the binary scale.
2. 4675 from the denary to the ternary scale.
3. 1765 from the denary to the septenary scale.
4. 1000000 from the denary to the duodenary scale.
5. 2332 from the quaternary to the quinary scale.
6. 10111011 from the binary scale to the ternary.
7. 54231 from the septenary scale to the undenary.
8. 10000 from the quinary scale to the nonary.
9. 1000000 from the duodenary scale to the senary.
10. 7777 from the octenary scale to the quaternary.

XIII.

1. Find the sum and difference of 2340 and 3214 in the quinary scale.

2. Find the product and quotient of 15321 and 132 in the septenary scale.

3. Find the sum, difference, product, and quotient of 743t and 1859 in the duodenary scale.

4. Find the quotient of 51117344 divided by 675 in the octenary scale, and reverse the process.

5. Divide 14332216 by 6541 in the nonary scale and multiply the quotient by 888.

6. Divide 100000 by 200, in the ternary, quaternary, and quinary scales, and express the sum of the quotients in the denary scale.

XIV.

1. Determine the weights which must be selected out of the series of 1, 2, 4, 8, &c., pounds each, respectively, in order to weigh 1000 pounds. Find also of the same series of weights which would be required to weigh 2487 pounds.

2. What weights must be selected out of the series 1, 3, 9, 27, 81, &c., pounds, to weigh 1907 pounds?

Find also which weights of the same series must be selected to weigh 716 pounds, not more than one of each weight being used.

3. State the three conditions required in a perfect system of arithmetical notation, and compare the advantages of the denary and duodenary scales of notation.

RESULTS, HINTS, ETC., FOR THE EXERCISES.

Notation and Numeration.

I.

1. 56 : 65. 2. 304 : 596. 3. 7707: 9273. 4. 27450. 5. 198567. 6. 5,500,505. 7. 95,059,000. 8. 710,710,710. 9. 1,000,100,000.

II.

1. Fifty-seven : seventy-five : eighty : forty-five : twenty-one : seventy-nine. 2. One hundred and one : three hundred : four hundred and twenty-five : six hundred and two : six hundred and twenty : nine hundred and ninety-nine. 3. One thousand and one : one thousand one hundred : one thousand and ten : four thousand seven hundred and eighty-three : two thousand nine hundred and nine : one thousand eight hundred. 4. Twenty-one thousand : twenty thousand one hundred : forty-eight thousand seven hundred and sixty-five : eighteen thousand and nine : eighteen thousand and ninety : ninety thousand and nine. 5. One hundred and twenty-three thousand four hundred and fifty-six : seven hundred thousand and nine : seven hundred and nine thousand : one hundred thousand one hundred : five hundred and five thousand and fifty. 6. Seven millions, six hundred and fifty-four thousand, three hundred and twenty-one : one million, two thousand and three : five million and eighty-one thousand : nine million, eighty thousand, seven hundred and six. 7. Twenty-nine million, three hundred and fifty-eight thousand, seven hundred and sixty-four : fifty million, eight hundred thousand, seven hundred : eighty million, fifty-three thousand and seven. 8. Five hundred and three million, one hundred thousand, nine hundred and fifty : twenty billion, one hundred million, ten thousand and ten. 9. One thousand billion, two hundred million, three thousand and one : five thousand seven hundred and eighty-six billion, seven hundred and thirty-two thousand, one hundred and ninety-five million, nine hundred and forty-six thousand, four hundred and seventy.

III.

2. One instance, the number 7, can be made up of 6 and 1, 5 and 2, 4 and 3, and in like manner the rest. 3. Proceed as in the last example. 4. 43210 is the largest, and 12340 is the smallest number. 5. One hundred thousand tens : ten thousand hundreds : one thousand thousands, ten tens of thousands. 6. First, when only one figure is employed there are 3 numbers ; when 2 figures 9, when 3 figures 18, and when 4 figures 18, making on the whole 48 different numbers.

Addition and Subtraction.

I.

1. 137. 2. 1,201. 3. 14,611. 4. 30,000. 5. 233431. 6. 1,429051. 7. 29,900000. 8. 5,213513. 9. 3,315931. 10. 6263,576043.

II.

1. See Art. 3. 2. See Art. 2. 3. See Art. 2. 4. Place 5 for the middle figure. 5. 24 different numbers can be formed with 4 figures.

III.

1. 18. 2. 19,001. 3. 445,545. 4. 907583. 5. 1,549830. 6. 76,543211. 7. 254,241,976. 8. 104,734222. 9. 2,922,930923.

IV.

1. The number is 85,018982. 2. 495799. 3. 846,055. 4. 4,876491,781190. 5. 944813. 6. 30576. 7. 229,495,326. 9. 49,999940,000050,060000.

V.

1. See Art. 5. 2. See Art 5. 4. 666269. 5. The numbers are 348 and 173. 6. 54365,636568.

VI.

1. Excess is 1131138. 2. 313618. 3. 32063. 4. 6693,878.

Multiplication and Division.

I.

1. Art. 1. 2. Art. 4. 3. Art. 4. 4. Art. 4. 5. Art. 2. 6. Art. 5 : Art. 4, note.

II.

1. 657152 : 7139954304 : 1261948567.
2. 12851451688656 : 987632423577.
3. 67500 : 3106800 : 3145200000.
4. 70997476000000 : 54839483410.
5. 659271431415. 6. 342704239630522424960.

III.

1. The multiplier 729819 is obviously composed of 729, 81 and 9, or 9, 9×9 and 9×81. If these three products of the multiplicand be found, and added together according to their local values, the amount will be the product obtained by 3 lines of multiplication instead of 6. A similar remark applies to the other 5 examples.

IV.

2. 87 times. 3. Multiply the given number by 20. 4. $648 = 640 + 8$. 5. $492 = 500 - 8$, multiply by 500 and by 8, and take the difference. 6. The quotients of 500000 and 1000000 by 4389 will give the limits. 7. See pp. 5, 6, Section I. 8. 86.

V.

1. Art. 1. 2. Art. 1. 3. See Art. 13. 4. Art. 9. 5. Art. 10.
6. $8 + 7 = 5 \times 3$, $10 - 4 = 30 \div 5$, or $\frac{30}{5}$.

VI.

1. Quot. 1872843, rem. 1: quot. 1248562, rem. 1: quot. 936421, rem. 3 : quot. 749137, rem. 2 : quot. 624281, rem. 1: quot. 535098, rem. 1: quot. 468210, rem. 7: quot. 416187, rem. 4.

2. Quot. 587668, rem. 46 : quot. 449944, rem. 62 : quot. 239970, rem. 78 : quot. 218155, rem. 18.

3. Quot. 424, rem. 596 : quot. 42, rem. 15956 : quot. 4, rem. 92756.

4. Quot. 418, rem. 300000 : quot. 168, rem. 8.

5. Quot. 999, rem. 1: quot. 139, rem. 119805.

6. Quot. 253, rem. 4807. 7. Quot. 777, rem. 2100.

8. Quot. 30939964, rem. 8 : quot. 2812723, rem. 98 : quot. 278738, rem. 413 : quot. 27848, rem. 7523 : quot. 2784, rem. 62459.

VII.

1. 10005100. 2. 19052. 3. Divisor 194, remainder 86. 4. 784623. 5. 450010000. 6. 10660526. 7. From the conditions stated, the remainder is 12, the divisor 72, and the quotient 432, whence the dividend is 31116.

VIII.

1. 8064. 2. 15625. 3. The factors of the multiplier are, 5, 7, 11, 12. 4. 235. 5. 7706307420. 6. 1343100 and 1379400. 7. The quotient is 10177314. 8. 272135. 9. 1428927.

IX.

2. 224. 3. The numbers are 256, 652, 526, 625, 562, 265, and it may be shewn that their sum can be divided by 13. 4. The product is 108 times 12. 5. Art. 5. 6. The numbers are 5003, 5009, 5011, 5017, 5021, 5023. 7. 144 is the number. 8. 60, 180, 300, and 420, are the parts. 9. Since the 1st part exceeds the 2nd, and the 2nd the 3rd by 88: the 1st exceeds the 3rd by 176. Hence 1st part is equal to 3rd part and 176, 3nd part is equal to 3rd part and 88: therefore the sum of all three parts is equal to 264 and 3 times the 3rd part. But the sum of the 3 parts is 1014. Hence it follows the 3 parts are 426, 338, and 250.

X.

1. The remainder is 17. 2. 174 times. 3. See Art. 11. 4. Art. 11. 5. The sum 1302, the difference 448. 7. 1379.

XI.

1. Exemplify by taking all the cases, 123, 234, 345, &c., and arranging the figures of each number in any order. It will be seen that all such numbers are multiples of 3.

2. Take all the numbers of six digits, each digit being the same figure, and verify the statement.

5. Exemplify this by taking all the possible cases.

7. The respective multiplicands are 8547, 17094, 25641, 34188, 42735, 52182, 59829, 68376, 76923.

XII.

1. 10110000. 2. 20102011. 3. 4612. 4. 41tt14. 5. 1221. 6. 20221. 7. t160. 8. 4252. 9. 144000000. 10. 333333.

XIII.

1. Sum 10104 : difference 324. 2. Product 2332022. 3. Sum 9097, difference 57t1, product 10688606. 4. Quotient 5726, remainder 346. 5. Quotient 2018, remainder 4837. Product of quotient by 888, 2015871. 6. The sum of the quotients in the denary scale is 432.

XIV.

1. 1000 in the denary scale is equal to 1111101000 in the binary $=2^3+2^5+2^6+2^7+2^8+2^9$, which gives the weights to be selected. Similarly 2847 requires the weights 1, 2, 2^2, 2^3, 2^4, 2^8, 2^9, 2^{11}.

2. 1907 when converted into the ternary scale, requires one of each of the weights 3^2, 3^3, 3^5, and two of each of the weights 1, 3, 3^4, 3^6.

When 716 converted into the ternary scale is equal to 222112,
or $2+3+3^2+2.3^3+2.3^4+2.3^5$, which may be reduced to
$-1-3-3^2+3^6$ by taking each 2 as equal to $3-1$.

Hence it will appear that if 3^6 be put into one scale, and 716 with the weights 1, 3, 3^2 into the other, they will balance each other.

3. Art. 14 and Art. 1 on Notation and Numeration.

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CONCRETE NUMBERS.

Art. 1. Quantity or magnitude comprehends whatever is capable of increase or decrease. The word is applied to all things in which a real distinction of units can be conceived to exist, and to those in which no such distinction naturally exists.

Discontinuous quantity includes all such things which consist of parts separate from each other, as a multitude of objects of any kind, and the number is known by ascertaining of how many objects the multitude consists.

Continuous quantity is that wherein there is no distinction of parts, or where the parts are all contiguous, or can be supposed to adhere together, as the line, the surface, the volume of space, of which any arbitrary portion of each may be assumed as the unit, and the exact number of units in any magnitude can be known by the number of times this assumed portion is contained in it. Hence it follows that all kinds of continuous quantity can be denoted by numbers.

Money.

2. The standard unit of English money is the pound sterling, which is divided into other smaller units, called shillings, pence, and farthings, according to the following table of money of account:—

4 farthings are equal to 1 penny.
12 pence ,, 1 shilling.
20 shillings ,, 1 pound.

The letters £, s, d, q, are the initial letters of the Latin words libra, solidus, denarius, quadrans, and were formerly used to denote the names of pounds, shillings, pence, and farthings. The letters £ s. d. are still employed for that purpose, but the q. is now seldom used, the fractions $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$ being employed to denote 1, 2, 3 farthings respectively. Since the four units of English money are the pound, the shilling, the penny, and the farthing, and that 20 shillings make one pound, 12 pence one shilling, and 4 farthings one penny; it follows that any number of pounds can be expressed in shillings, by multiplying the number of pounds by 20; shillings can be expressed in pence, by multiplying the number by 12; and pence in farthings, by multiplying the number by 4.

And conversely. Any number of farthings can be changed to pence by dividing the number of farthings by 4, pence to shillings by dividing the number by 12, and shillings to pounds by dividing the number by 20.[1]

3. Prob.—*To find the sum and the difference of sums of money.*

In the processes of addition and subtraction of money expressed in pounds, shillings, pence, and farthings, the same principle is employed as in the addition and subtraction of abstract numbers, with this difference. In the latter, 10 units of any order make 1 of the next superior order; but in the units of English money, 4 of the smallest units make 1 of the second or next greater; 12 of the second units make 1 of the third; and 20 of the third units make 1 of the fourth or the greatest unit. Hence in adding the different units of money beginning with the smallest; for every 4 farthings, 1 must be added to the pence; for every 12 of the pence, 1 must be added to the shillings; and for every 20 of the shillings, 1 must be added to the pounds.[2] The same must be observed also in finding the difference of two sums of money.[1]

---

[1] Ex. To find how many farthings there are in £16 14s. 9¼d.

£ s. d.
16 14 9¼
20 shillings in one pound.
334 shillings in £16 14s.
12 pence in one shilling.
4017 pence in £16 14s. 9d.
4 farthings in one penny.
16069 farthings in £16 14s. 9¼d.

Conversely. How many pounds, shillings, pence, and farthings are there in 16069 farthings?

4)16069 farthings.
12) 4017 pence and 1 farthing over, or ¼.
20) 334 shillings, and 9 pence over.
16 pounds and 14 shillings over.

Hence 16069 farthings are equal to £16 14s. 9¼d.

[2] Ex. To find the sum of £53 12s. 8½d. and £25 15s. 9¾d.

£ s. d.
53 12 8½
25 15 9¾
£79 8 6¼ sum.

The process may be thus described:

First. 3 farthings and 2 farthings are 5 farthings, which are equal to 1 penny and ¼; reserve the ¼d.

Secondly. 1 penny, 9 pence, and 8 pence are 18 pence, which are equal to 1 shilling and 6 pence; reserve the 6d.

Thirdly. 1 shilling, 15 shillings, and 12 shillings are 28 shillings, which are equal to 1 pound and 8 shillings; reserve the 8s.

Fourthly. 1 pound, 5 pounds, and 3 pounds are 9 pounds.

Lastly. 20 pounds and 50 pounds are 70 pounds.

Hence, the sum is 79 pounds 8 shillings 6 pence and 1 farthing.

The process of the addition of money may be facilitated by committing to memory the numbers of shillings and pence in all the numbers of pence between 12 and 200, as also the number of pounds and shillings in all the numbers of shillings between 20 and 200.

The object of multiplication is to find the sum or product by a short and simple process which will render repeated additions unnecessary. The definition is strictly applicable to all abstract numbers, and to concrete numbers when the multiplier is an abstract number. In all other cases, the meaning of the product must be modified to suit particular cases, and at the same time made to coincide with the definition in all points which the factors possess in common.

It has been seen (Sect. VI. Art. 2.) that the product of two abstract numbers, 3 and 5, is 15, an abstract number consisting of abstract units. And conversely, if 15 be divided by 3, the quotient is 5, an abstract number. If, however, 5 denote 5s. a concrete number, then the product 5s. × 3 = 15s. a concrete number; and conversely, 15s. ÷ 3 = 5s. a concrete number.[2] But 15s. ÷ 5s. = 3, an abstract number, denoting that 5s. is contained 3 times in 15s. Also if 3 denote

---

[1] Ex. To find the difference between £79 8s. $6\frac{1}{4}$d. and £53 12s. $9\frac{1}{2}$d.

£	s.	d.	
79	8	$6\frac{1}{4}$	
53	12	$9\frac{1}{2}$	
£25	15	$8\frac{3}{4}$	difference.

First. 2 farthings cannot be taken from 1 farthing: add 4 farthings to 1 farthing, then 2 farthings taken from 5 farthings leave 3 farthings; reserve the $\frac{3}{4}$d.

Secondly. 1 penny added to 9 pence make 10 pence; which cannot be taken from 6 pence: add 12 pence to 6 pence, then 10 pence taken from 18 pence leave 8 pence; reserve the 8d.

Thirdly. 1 shilling added to 12 shillings make 13 shillings, but 13 shillings cannot be taken from 8 shillings: add 20 shillings to 8 shillings, then 13 shillings taken from 28 shillings leave 15 shillings; reserve the 15s.

Fourthly. 1 pound added to 3 pounds make 4 pounds, and 4 pounds taken from 9 pounds leave 5 pounds.

Lastly. 50 pounds taken from 70 pounds leave 20 pounds.

Hence the difference is 25 pounds 15 shillings 8 pence and 3 farthings.

The use of the addition and subtraction of money may be illustrated in various ways. For example: In an account of the transactions in money between two persons, where receipts and payments are made between them. Each person in stating his account would consider himself *debtor* for what he receives from, and *creditor* for what he pays to, the other. To ascertain at any time the real state of the account between them, the sums *received* would be added together, and also the sums *paid*, and the smaller sum taken from the larger leaves a remainder. If the sum received as *creditor* be the greater, the difference is due to the *debtor*, so as to make the balance of the two accounts equal. If the sum paid as *debtor* be the greater, the difference is due to the *creditor* to make the balance of their accounts equal.

[2] If 120 shillings be divided by 20, an abstract number, the quotient is 6 shillings, the twentieth part of 120 shillings. If 120 shillings be divided by 20 shillings, the quotient is 6, a number indicating that 20 shillings is contained 6 times in 120 shillings. But as 20 shillings make the larger unit of 1 pound, the number 6 will denote the number of pounds contained in 120 shillings.

3s., then the product is 3s. × 5 = 15s., the same as before; and conversely 15s. ÷ 5 = 3s.; also 15s. ÷ 3s. = 5, an abstract number, denoting that 3s. is contained 5 times in 15s.

Hence since 5s. × 3 = 15s. and 3s. × 5 = 15s., it follows that in the product of two factors, one of which is concrete and the other abstract, the abstract and concrete factor may be interchanged without affecting the meaning of the product.[1]

If both factors be concrete of the same kind, as 3s. and 5s., the product 3s. × 5s. appears not to admit of an intelligible meaning, unless 3s. and 5s. each denote 3 and 5 pieces of coin. In that case, 5 shilling coins may be considered as 5 times 1s., and 3 shilling coins 3 times 1s., and the product 3s. × 5s. may be taken to mean 5 times 3 times 1s. or 15 shillings, which admit of being arranged in 3 rows, with 5 in each row; or in 5 rows with 3 in each row.

If 3 and 5 denote units of different kinds, as 3s. and 5 yards, the product, 3s. × 5 yards, is apparently absurd.[2] But 5 yards is nothing more than 5 times 1 yard; and since 3s. corresponds to 1 yard, if 3s. be taken 5 times, or 3s. be multiplied by 5, the result, 15s., corresponds to 5 yards, which affords an intelligible meaning to the product.

And conversely, 15s. ÷ 3 yards may be explained in the same manner as giving the quotient 3s. In the same manner other products and quotients of two concrete factors of the same or different units may be considered and interpreted.

4. Prob. *To multiply and divide any sum of money by an abstract number.*

1. When the multiplier consists of a single figure.

The product of a sum of money by an abstract number can be found by taking the several units of farthings, pence, shillings, pounds as many times as the multiplier denotes, considering at each step how many of any number of units make one of the next greater, and beginning with the smallest unit.[3]

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[1] The interchange of the abstract units for the concrete, or the concrete for the abstract in the product of two factors, can sometimes be made with great advantage in the performance of computations.

[2] The ordinary forms in which such multiplications and divisions occur are in such questions as the following:—If one yard of cloth cost 3s., what is the cost of 5 yards? and, If 5 yards of cloth cost 15s., what is the cost of one yard?

In the former question there is given the fact of 3 shillings corresponding to 1 yard, to find the number of shillings which correspond to 5 yards.

In the second question there is given the fact of 15 shillings corresponding to 5 yards, to find the number of shillings which correspond to 1 yard.

[3] Ex. To multiply £5 14s. $7\frac{3}{4}$d. by 6.

£	s.	d.	
5	14	$7\frac{3}{4}$	
		6	
£34	7	$10\frac{1}{2}$	product.

Conversely. A sum of money can be divided by an abstract number by beginning with the largest units, and dividing each in succession, observing how many smaller units make one of the next greater.[1]

First. After the pounds of the quotient are found, let the remaining pounds be reduced to shillings.

Secondly. When the shillings of the next partial quotient are determined, let the remaining shillings be reduced to pence.

Thirdly. When the pence of the next partial quotient are found, let the remaining pence be reduced to farthings.

Fourthly. Let the farthings of the last partial quotient be found.

Then the complete quotient will consist of those partial quotients, and will in general be composed of pounds, shillings, pence, and farthings.

2. When the multiplier is a large number, it will sometimes be found convenient to find separately the products of the different units of the multiplicand by the multiplier, making a change of multiplier and multiplicand in each case, and reducing each product when

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The process is exactly the same as that of multiplying any number by another consisting of one figure (Art. 3, Sect. VI.), with this difference, that at each step of the operation, attention must be paid to the number of smaller units which make one of the next greater.

First. 3 farthings multiplied by 6 gives 18 farthings, or 4d. and $\frac{1}{2}$; reserve the $\frac{1}{2}$d.

Secondly. 7d. multiplied by 6 with 4d. added make 46d., or 3s. and 10d.; reserve the 10d.

Thirdly. 14s. multiplied by 6 with 3s. added make 87s., or £4 and 7s.; reserve the 7s.

Fourthly. £5 multiplied by 6 with £4 added make £34.

The product is £34 7s. $10\frac{1}{2}$d.

[1] Ex. To divide £34 7s. $10\frac{1}{2}$d. by 6.

$$\begin{array}{rrrl} & £ & \text{s.} & \text{d.} \\ 6) & 34 & 7 & 10\frac{1}{2} \\ \hline & £5 & 14 & 7\frac{3}{4} \text{ quotient.} \end{array}$$

The process of division being the reverse of multiplication, is (in this case) analogous to that of the division of any number by another of one figure, with the difference above stated. In division, the operation must begin with the greatest units and end with the least.

First. £34 divided by 6 gives the quotient £5, and £4 or 80s. over; reserve the £5.

Secondly. 80s. added to 7s. make 87s., which divided by 6 gives the quotient 14s., and 3s. or 36d. over; reserve the 14s.

Thirdly. 36d. added to 10d. make 46d., which divided by 6 gives the quotient 7d., and 4d. or 16 farthings over; reserve the 7d.

Fourthly. 16 farthings added to 2 farthings make 18 farthings, which divided by 6 gives the quotient 3 farthings.

The quotient is £5 14s. $7\frac{3}{4}$d, the sum of the partial quotients.

necessary, to pounds, &c., and taking the amount of the product so found.[1]

By a second process the product of a sum of money by a large number may be known by finding 10, 100, 1000, &c., times the given sum, and then multiplying each of these products respectively by the given number of units, tens, hundreds, thousands, &c., in the multiplier, and taking the amount of these products.[2]

By another method analogous to that exhibited (Sect. VI. Art. 4) the product can be found by multiplying the given sum successively by any factors into which the multiplier can be divided, either complete or incomplete.[3]

Conversely. The quotient arising from the division of a sum of money by a large abstract number can always be determined in a

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[1] Ex. Multiply £5 14s. $7\frac{3}{4}$d. by 2136.

			£	s.	d.
First,	£5 × 2136 = £2136 × 5	=	10680	0	0
Next,	14s. × 2136 = 2136s. × 14	=	1495	4	0
Thirdly,	7d. × 2136 = 2136d. × 7	=	62	6	0
Fourthly,	$\frac{3}{4}$ × 2136 = 2136q. × 3	=	6	13	6
Product of £5 14s. $7\frac{3}{4}$d. by 2136		=	£12244	3	6

[2] Ex. Multiply £5 14s. $7\frac{3}{4}$d. by 2136.

Here 2136 = 2000 + 100 + 30 + 6.

£	s.	d.		
5	14	$7\frac{3}{4}$		
		10		
57	6	$5\frac{1}{2}$ =	10 times £5 14s. $7\frac{3}{4}$d.	
		10		
573	4	7 =	100 times	,,
		10		
5732	5	10 =	1000 times	,,
		2		
11464	11	8 =	2000 times	,,
573	4	7 =	100 times	,,
171	19	$4\frac{1}{2}$ =	30 times	,,
34	7	$10\frac{1}{2}$ =	6 times	,,
12244	3	6 =	2136 times £5 14s. $7\frac{3}{4}$d.	

[3] Ex. Multiply £5 14s. $7\frac{3}{4}$d. by 48 and by 51.

Here 48 = 8 × 6; and 51 = 8 × 6 + 3.

£	s.	d.		
5	14	$7\frac{3}{4}$		
		8		
45	17	2 =	8 times £5 14s. $7\frac{3}{4}$d.	
		6		
275	3	0 =	48 times	,,
17	3	$11\frac{1}{4}$ =	3 times	,,
292	5	$11\frac{1}{4}$ =	51 times	,,

manner analogous to that in Sect. VI. Art. 9, regard being had to the units of magnitude.[1]

If the divisor can be exactly divided into small factors, the quotient of a sum of money may be found by dividing the given sum successively by the factors of the divisor.[2]

IMPERIAL WEIGHTS.

5. By the Act of Parliament which defined the standards of weights and measures, it was enacted, that "a cubic inch of distilled water, weighed in air by brass weights, at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer being at 30 inches, is equal to 252·458 grains."

Of these grains thus determined the pound Avoirdupois consists of 7,000.

AVOIRDUPOIS WEIGHT.

16 drams are equal to 1 ounce
16 ounces ,, 1 pound
28 pounds ,, 1 quarter
4 quarters ,, 1 hundredweight
20 hundredweights ,, 1 ton.

Avoirdupois weight is the general weight employed in weighing all substances except gold, silver, and precious stones.

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[1] Ex. Divide £12244 3s. 6d. by 2136.

£ s. d.
2136)12244 3 6(£5 14s. $7\frac{3}{4}$ quotient.
10680
1564
20
2136)31283(14s.
2136
9923
8544
1379
12
2136)16554(7d.
14952
1602
4
2136)6408(3q.
6408
0

[2] Ex. Divide £275 3s. by 48.

Here $48 = 6 \times 8$.
£ s. d.
6)275 3 0
8)45 17 2
£5 14 $7\frac{3}{4}$ quotient.

APOTHECARIES' LIQUID MEASURE.

60 minims	are equal to	1 dram
8 drams	,,	1 ounce
20 ounces	,,	1 pint.

The liquid dram and ounce are the same as the Avoirdupois dram and ounce.

TROY WEIGHT.

24 grains	are equal to	1 pennyweight
20 pennyweights	,,	1 ounce
12 ounces	,,	1 pound.

Troy weight is used for gold and silver and precious stones.

English standard gold contains 22 parts pure gold and 2 parts alloy, and is named by goldsmiths, gold of 22 carats fine. The price of standard gold is fixed by law at £3 17s. 10½d. an ounce, but the Mint purchases gold at £3 17s. 9d. an ounce, with certain small allowances.

Standard silver consists of 37 parts pure silver and 3 parts alloy.

MEASURES OF LENGTH, SURFACE, AND VOLUME.

6. There are three distinct kinds of magnitude, the line or length, the surface or area, and the volume or solid. The line has one dimension, length only. The surface has two dimensions, length and breadth. The solid has three dimensions, length, breadth, and thickness.

The unit of length is a straight line of definite and known length, assumed as the primary unit of measure. Any length or distance is measured by finding how many times the assumed unit of length is contained in the distance to be measured. Units both of greater and less magnitude have been contrived for the measurement of large and small distances, and in general consist of some multiple or submultiple of the assumed primary unit of length.

The square is assumed for the measure of surfaces, and the unit of area is assumed to be a square the side of which is a line one unit in length. To measure a surface or area is to determine the number of times it contains the square unit of area.

To shew that the number of square units in a rectangle or a rectangular parallelogram is expressed by the product of the numbers of lineal units contained in the length and breadth of the rectangle.

Let the side AB of the rectangle $ABCD$ contain 5 units and the breadth AD 4 units.[1]

The product 5×4 will denote the number of square units in the area of the rectangle.

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[1] The student is recommended to draw the figures for himself. He may see the figures referred to in the notes on Books II. and XI. of the Editor's Edition of Euclid's Elements.

Let lines parallel to the sides be drawn through the points of division of two adjacent sides.

Then the side *AB* contains 5, and the side *AD* 4 lineal units.

And since for every lineal unit in *AD*, there are 5 square units on *AB*, it follows that there are 4 × 5 square units in the rectangle *ABCD*. Or in other words, the product of two numbers representing straight lines is equal to a number representing an area.

Hence it follows, that if a number representing an area be divided by a number representing a line, the quotient will be a number representing a line.

The cube is the figure assumed for the measure of volumes or solids. The unit of volume is assumed to be a cube whose edges are each one unit in length, and to measure a volume is to ascertain how many times the volume contains the unit of volume.

To shew that the number of cubic units contained in the volume of a rectangular parallelopiped is expressed by the product of the three numbers of the lineal units contained in its three adjacent edges.

Let *AB, AE, AD,* three adjacent edges of the figure, contain 5, 4, 3 lineal units respectively. Then 5 × 4 × 3 will denote the number of cubic units in the volume.

For if planes be drawn parallel to the sides through the points of division of three adjacent edges, the whole volume will be divided into cubic units equal to one another.

Now the surface *ABCD* contains 5 × 3 square units; and for every linear unit in *AE*, there is a layer of 5 × 3 cubic units corresponding to it; and since there are 4 lineal units in *AE*, consequently there are 4 × 5 × 3 cubic units in the volume of the parallelopiped.

And the product of the numbers of lineal units in any three adjacent edges denotes the number of cubic inches in the volume. Hence it follows that the product of three numbers representing straight lines will denote the number of cubic units in that rectangular parallelopiped of which the numbers denote the lengths of three adjacent edges.

Hence, if a number representing a volume be divided by a number representing a line, the quotient will be a number representing a surface; and

If a number representing a volume be divided by a number representing a surface, the quotient will be a number representing a line.

It may also be noted, that,

If a number representing a line, a surface, or a volume, be each multiplied by an abstract number, the product will in each case represent a line, a surface, or a volume respectively.

And conversely. If a number representing a line, a surface, or a volume, be respectively divided by another number representing a line, a surface, or a volume, the quotient in each case will be an abstract number.

In these conclusions, the lineal unit, as also the square unit and the cubic unit, though limited in magnitude, are not restricted to any standard unit, but may be any multiple or submultiple of any standard unit whatever. These units may be increased or diminished in any degree, and consequently, the products which express the square units in any area, and the cubic units in any volume, may be denoted either by fractional or decimal numbers.

The Act for Imperial Weights and Measures, passed on 17th June, 1824, has declared that a certain brass rod is the standard yard of England. That its length, compared with that of the seconds pendulum at the level of the sea, is in the proportion of 36 to 39·1393 inches, the latter being the exact length of the seconds pendulum in a vacuum in the latitude of London.

MEASURES OF LENGTH.

3	barleycorns are equal to		1 inch
12	inches	,,	1 foot
3	feet	,,	1 yard
5½	yards	,,	1 rod, pole, or perch
40	poles	,,	1 furlong
8	furlongs	,,	1 mile
69½	or 60 geographical miles		1 degree.

There are other measures employed in different trades and employments, as the palm of 3 inches, and the hand of 4 inches, the cubit of a foot and a half, the fathom of 6 feet, and the larger unit of a league of 3 miles.

Engineers divide the inches into tenths as a more convenient division for small portions than eighths or twelfths.

The chain devised by Mr. Gunter, and called after his name, is 22 yards long, and consists of 100 links, so that 80 chains are equal to one mile. The practical use of this chain, both for measuring lengths and surfaces, led to its adoption, and it has continued in use both for the measuring of distances and surfaces, 10 square chains being equal to one acre.

MEASURES OF SURFACE.

144	square inches are equal to		1 square foot
9	square feet	,,	1 square yard
30¼	square yards	,,	1 square rod, pole, or perch
40	poles	,,	1 rood
4	roods	,,	1 acre
640	acres	,,	1 square mile.

Paper is measured by considering 24 sheets to one quire, and 20 quires to a ream. The ream of 20 quires is called the mill ream, and is different from the printer's ream, which consists of one quire and a half more than the mill ream. These numbers do not describe the size of the sheets of paper, which are of various sizes and different names.

Cloth Measures.

2¼ inches are equal to 1 nail
4 nails ,, 1 quarter
4 quarters ,, 1 yard.

The ell was used formerly as a measure; the English ell consisted of 5 quarters, the French ell of 6 quarters, and the Flemish ell of 3 quarters. As cloths are made of different widths, it is not customary to measure cloth by the number of square yards, but only by the lengths; the widths, however, are taken into consideration in the value of the length of a yard, as one yard of 6 quarters wide would be double the value of a yard of 3 quarters wide of the same kind of cloth.

Measures of Volumes or Solids.

1728 cubic inches are equal to 1 cubic foot
27 cubic feet ,, 1 cubic yard.

There are other measures employed for the measurement of different kinds of merchandise.

A cubic foot of water is generally estimated at 1,000 ounces, which are nearly equal to 62½lbs. Avoirdupois.

Imperial Measures of Capacity.

7. The standard unit of the measure of capacity defined by the Act for the Uniformity of Weights and Measures is the imperial gallon, containing 10 pounds Avoirdupois of distilled water at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer standing at 30 inches, and equal in capacity to 277·274 cubic inches.

The gill is defined to consist of 8·665 cubic inches. The Act ordered the imperial gallon to be adopted in all measures instead of the old gallon, and directed this measure to be used for dry and liquid articles of merchandise.

4 gills are equal to 1 pint
2 pints ,, 1 quart
4 quarts ,, 1 gallon.

Any measure which is a multiple of the gallon is permissible by the Act for imperial measures. The ordinary measures in use at the time of the passing of the Act became legal if the gallon employed was the imperial gallon.

The new imperial gallon is used for dry substances—corn, seeds, &c.

2 gallons are equal to 1 peck
4 pecks ,, 1 bushel
4 bushels ,, 1 coomb
2 coombs ,, 1 quarter
4 quarters ,, 1 chaldron.

Different measures are employed in different trades. Corn is generally regulated by the quarter. Coals used to be measured by the sack of 3 bushels, and the chaldron of 12 sacks; they are now measured by weight, the ton being employed as the unit of weight.

Measures of Liquid Substances.

Ale, Beer, &c.

2 pints are equal to 1 quart
4 quarts ,, 1 gallon
9 gallons ,, 1 firkin
2 firkins ,, 1 kilderkin
2 kilderkins ,, 1 barrel
3 kilderkins ,, 1 hogshead
2 hogsheads ,, 1 butt.

Wines, Spirits, &c.

2 pints are equal to 1 quart
4 quarts ,, 1 gallon
63 gallons ,, 1 hogshead
2 hogsheads ,, 1 pipe
2 pipes ,, 1 tun.

There were other measures of wine in use, such as the anker of 10 gallons, the runlet of 18, the tierce of 42, and the puncheon of 84.

Any quantity expressed in terms of a larger unit of measure, can also be expressed in terms of a smaller unit which is some subdivision of that measure, and conversely. The units of larger magnitude are changed to units of smaller magnitude of the same kind by multiplying the number of larger units by the number of smaller which make one of the larger; and conversely, the units of smaller magnitude are changed to those of a larger of the same kind by dividing the smaller by as many of the smaller as make one of the larger.

Angular and Circular Measures.

8. The invention of the sexagesimal division of the circumference of the circle is attributed to Ptolemy by his commentator Theon, but it is clear from the language of Ptolemy himself that this mode of division was known long before his time. It is more probable that the improvement made by Ptolemy was the extension of that mode of division to the radius or the chord of a circle which subtends an arc equal to one-sixth of the whole circumference.

It has been conjectured that the division of the circumference of a circle into 360 equal parts was suggested by the fact that the sun appears to move once round the whole circle of the heaven in the period of about 360 days; but whether this or the converse be the truth, it is quite impossible to affirm. This division of the circumference has come down from very early times.

Angles are measured by the arcs of circles. If a circle be described with any radius from the angular point as a centre, the arc intercepted between the lines which contain the angle may be assumed to be the measure of the angle; for the angle increases or decreases as the arc which subtends it at the center increases or decreases.

If in a circle two diameters be drawn at right angles to each other, the circumference is divided into four equal parts, called quadrants, and each of these quadrants subtends a right angle at the centre of the circle. If any one of the quadrants be divided into ninety equal arcs, and straight lines be drawn from the centre of the circle to all the divisions of the quadrant, it will be seen that each of the ninety arcs subtends ninety equal angles at the centre of the circle. The angle which is the ninetieth part of a right angle is called a *degree*, and the ninetieth part of the quadrantal arc is also called *a degree* of the quadrant.

The *angular degree* and the *circular degree* must not be confounded, as the former is the ninetieth part of a right angle, and the latter is the small arc which subtends that small angle at the centre of the circle. Each degree is divided into sixty equal parts called minutes, and each minute into sixty equal parts called seconds, and so on, according to the following table:

60 seconds are equal to 1 minute.
60 minutes ,, 1 degree.
90 degrees ,, 1 quadrant, or a right angle.
4 quadrants ,, 1 circumference of a circle, or 4 right angles.

It may also be observed, that the radius being equal to the chord of the arc which subtends an angle of sixty degrees, it is not improbable this fact may have suggested the sexagesimal division of the degree, the minute, the second, &c. It may also be noted, that the whole circumference of a circle can be divided into six equal arcs, each arc subtended by a chord equal to the radius, and each of these arcs will subtend an angle of sixty degrees at the centre of the circle.

There is another division[1] of the circumference of the circle, which indicates the points of the mariner's compass. This division is rather inaccurately named "points of the compass," which expression more fitly denotes the directions to which the needle points than the angular space between two successive divisions. The cardinal points of the compass are the east, west, north, and south. The east is reckoned from the centre (on which the needle moves) towards the right, the west towards the left of the horizontal diameter, the north towards the upper part, and the south towards the lower part of the other diameter at right angles to the former.

The whole circumference is divided into 32 equal arcs, of which each quadrant contains 8, and the length of each is 11° 15′.

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[1] This division is easily made. Describe a square, join the bisections of the opposite sides, draw the diagonals, and inscribe a circle in the square. The circumference of the circle is divided into 8 equal arcs, next dividing each of these arcs into 4 equal arcs, the 32 points are found, and can be marked beginning at E, W, N, or S.

The eight points in each of the four quadrants are expressed by means of the initial letters of the four cardinal points. Beginning at the north, the following are the successive points in the quadrant between the north and east points:—N., N. by E., N.N.E., N.E. by N., N.E., N.E. by E., E.N.E., E. by N.; next, the points between the east and south points, E., E. by S., E.S.E., S.E. by E., S.E., S.E. by S., S.S.E., S. by E.; thirdly, the points between the south and west points, S., S. by W., S.S.W., S.W. by S., S.W., S.W. by W., W.S.W., W. by S.; fourthly, the points between the west and north points, W., W. by N., W.N.W., N.W. by W., N.W., N.W. by N., N.N.W., N. by W.

Measures of Time.

9. The following table exhibits the ordinary divisions of time:

60 seconds	are equal to	1 minute.
60 minutes	,,	1 hour.
24 hours	,,	1 day.
7 days	,,	1 week.
4 weeks	,,	1 month.
12 months	,,	1 year.
or 365¼ days	,,	1 Julian year.

Common years consist of 365 days, and leap years, which happen every four years, consist of 366 days. The calendar months, January, March, May, July, August, October, December, consist each of thirty-one days. The months of April, June, September, and November, each of thirty days, and the month of February of twenty-eight days, except in leap years, when it consists of twenty-nine days. This year happens every fourth year, with the omission of three in every period of 400 years. The sexagesimal division of the hour, the minute, &c., most probably had its origin in the sexagesimal division of the circumference of the circle.

If the dividend denote the units of effect produced by an agent in a given number of units of time, then the quotient arising from dividing the units of effect by the units of time will denote the effect produced in one unit of time.

If the dividend denote units of distance and the divisor units of time, the quotient will denote the number of units of distance which correspond to one unit of time. As if a body move over uniformly 100 feet in four seconds of time, then the quotient, twenty-five feet, is moved over in one second of time. The distance moved over by a body in an unit of time is called the velocity of the body.

If the dividend denote units of distance, and the divisor denote the distance moved over in one unit of time, the quotient will denote the number of units of time in which the body will move over that distance. If a body move over 100 feet at the rate of twenty-five feet in one second of time, then the quotient of 100÷25 denotes four seconds, the time in which the body moves over 100 feet.

EXERCISES.

I.

1. Reduce £15 16s. $8\frac{1}{2}$d. to farthings, reverse the process, and explain the reason of each step.

2. How many fourpenny pieces must be given in exchange for half-a-guinea, half-a-sovereign, and half-a-crown?

3. How many half-crowns, how many sixpences, and how many fourpences are there in 200 guineas?

4. How many half-crowns, and how many half-guineas are there in a million of fourpenny pieces?

5. How many farthings are there in five half-sovereigns, five half-crowns, five sixpences, and five halfpence?

6. Reduce £13 to shillings, £13 15s. to pence, £13 15s. 7d. to farthings, and reverse the operations.

7. In the reduction of farthings to pounds, the farthings are first reduced to pence by dividing by 4; the pence to shillings by dividing by 12; and the shillings to pounds by dividing by 20; in what case would the same result be obtained by dividing the farthings, first by 20, next by 12, and lastly by 4?

II.

1. Find the amount of £25 10s. 6d., £120 11s. $4\frac{1}{2}$d., £247 19s. $9\frac{1}{4}$d., £1,000, and £1 17s. $11\frac{3}{4}$d.

2. Subtract £44 17s. $10\frac{3}{4}$d. from £100 0s. $0\frac{1}{4}$d., and verify the truth of the result.

3. What is the amount in pounds sterling of 25695426 shillings, 632985 florins, 2984695 crowns, and 3296 half-sovereigns, and what is the number of pieces of money?

4. In an account current between two persons, *A* and *B*, at the end of the first quarter *A* owed *B* £127 10s. $7\frac{1}{2}$d., at the end of the second *B* was in debt to *A* £100 11s. $7\frac{3}{4}$d., and at the end of the third quarter he was further in debt to *A* £95 17s. $8\frac{1}{2}$d.; but at the end of the fourth quarter *A* was in debt to *B* £190 19s. $11\frac{1}{2}$d. What was the balance at the end of the year, and to which of them was it due?

5. The total value of British goods exported from Liverpool during the 9 months ending October 31, in the years 1865, 1866, 1867, were £59,502,204, £73,967,069, and £63,960,580 respectively; find the increase of the value of the goods exported in 1867 upon the value in 1865, and the decrease upon the value in 1866.

III.

1. Multiply £11 11s. 11¾d. by each of the nine digits respectively, and reverse the operation in each case.

2. Multiply £14 17s. 11¼d. by 35, 37, and 175 respectively.

3. Multiply £963 1s. 1¾d. by 999.

4. Multiply £10 17s. 6¾d. by 8764.

5. Divide £192 15s. 11d. by 13, and establish the truth of the process.

6. Divide £20,000 19s. 6d. by 57621.

7. Divide 18s. 4d. by 44 and by 3s. 8d., and explain the nature of the quotients.

8. Find how often £24 11s. 6¼d. is contained in £8060 18s. 10d.

9. What is the difference between 5482 times £1 12s. 6½d. and 1562 times £5 14s. 2½d.?

10. What is the standard unit of English money? Explain under what conditions one concrete number may be multiplied by another.

IV.

1. What sum will be required to pay 18s. 6d. to each of 26 persons?

2. Among how many persons must £59 6s. 3¾d. be divided, that each person may receive £3 2s. 5¼d.?

3. Among how many persons can £1482 18s. 4½d. be divided, so that each may receive £12 1s. 1¼d.?

4. Find the price of 7865 articles at £2 19s. 6d. each, using one multiplication and one division.

5. Required the value of 2710 pounds at 6s. 8d. a pound.

6. If 6 articles cost £3 16s. 6d., how much will 13 cost at the same rate?

7. How much a day is £82 10s. 1¼d. a year?

8. Divide £1000 equally among 121 persons; and if the silver coins of the value of threepence be employed, what is the value of the remainder?

9. There are three quantities, £5, 8s., and 75 gallons. Multiply one of them by the quotient of the other two. State accurately the result of the operation, and perform it in as many different ways as possible.

V.

1. How many crowns are equal in value to 200 half-guineas?

2. If an English guinea be worth 27½ French francs, at 10d. each, how much French money ought a person to receive for 44 guineas?

3. How many coins of 5s. 0¼d. are worth 59527 coins of 5s. 1¾d. each?

4. An Austrian souverain and a gold ducat are worth 13s. 11d. and 9s. 6d. respectively. How many ducats are equivalent to 4560 souverains?

5. The value of a mark being 13s. 4d., and that of a moidore 27s., how many half-crowns are there in 30 marks and 40 moidores together?

6. How many roubles, at 3s. $4\frac{1}{2}$d. each, are worth 132678 francs at $9\frac{3}{4}$d. each?

7. If nine rupees, six crown pieces, and eleven threepenny pieces amount to £2 13s., what is the value of a rupee?

8. How many pounds sterling amount to 274 marks, each equal to 13s. 4d., and 87 nobles, each equal to 6s. 8d.?

VI.

1. A debt of 20 guineas is to be discharged by three equal payments in half-crowns, shillings, and sixpences; find the number of coins in each payment.

2. Divide £34 4s. into two parts, so that the number of crowns in the one may be equal to the number of shillings in the other.

3. A debt of £18 15s. 10d. is paid in crowns, shillings, and pennies, giving for every penny two shillings and three crowns; find the number of each sort of coins.

4. *A* has as many sovereigns as *B* has shillings, and together they have £210; required the number each had.

5. One person has a certain number of shillings, another as many crowns, and a third as many pounds, and together they have £520. How much has each?

6. Divide £34 into an equal number of sovereigns, half-sovereigns, half-crowns, shillings, and sixpences.

7. The sum of £211 10s. 5d. was paid in pounds, shillings, and pence, the number of pounds being 7 times the number of shillings, and the number of shillings 6 times the number of pence. How many were there of each?

8. A bag contains sixpences, shillings, and half-crowns; the three sums of money expressed by the different coins is the same; if there are 102 coins in the bag, find the number of sixpences, shillings, and half-crowns.

VII.

1. Show that one penny a day is equal to a sovereign, half-sovereign, and five pence a year; and hence deduce a rule for finding how much any daily payment would amount to in one year.

2. A person's income is reduced from £750 to £734 7s. 6d. by income tax. How much does he pay in the pound?

3. When the income tax was 4d. in the pound a person paid £2 18s. 5d., and when it was 6d. in the pound he paid £3 13s. 7d. What was the difference between his income in the two years?

4. If a person's daily expenditure be £2 12s. 6d. what must be his annual income in order that he may save £250 a year?

5. How much can be saved out of an annual income of £706 a year, if the daily expenditure be a guinea and a half?

6. If an artificer's income be £7 a week, what may be his daily expenditure in order that he may save yearly £100?

7. A labourer receiving wages at the rate of 15s. a week through the year, saved 6d. a week, which he deposited in the savings bank. In consequence of a strike his wages were raised 2s. a week, but he found at the same time that with the cost of living, and his payments to the trades' union, he had to pay 1s. 3d. where he formerly paid 1s.: find the increase made in his yearly savings.

8. A person's quarterly income is £135 10s., and his daily expenditure £2 5s., how much will he be in debt at the end of two years and a half?

9. If a person spend in 4 months as much as he gains in 3, how much will he have saved at the end of three years, supposing his half-yearly income is £160 10s.? What would have been the state of his affairs if his expenditure had been £160 10s. in 6 months?

10. If a person's gross income be £1,250 a year, what is his clear income after paying an income tax of 7 pence in the pound? If his clear income were £1,250 after paying the same income tax, what would be his gross income?

VIII.

1. If the price in shillings of a hundred weight be multiplied by 3 and divided by 7, the result is the value in farthings of a pound weight.

2. If 2613 pounds of sugar at 7d. a pound be exchanged against 417 pounds of tea at 4s. 6d. a pound, on which side is the balance, and how much?

3. How much tea at 3s. 6d. a pound must be given in exchange for 5cwt. 2qrs. of sugar at 4½d. per pound?

4. If 180 oranges be bought for 2 a penny, and 180 more at 3 a penny, what is gained or lost by selling the whole at 5 for 2 pence?

5. A merchant exchanged 1,200lbs. of pepper at 13d. a pound, for equal quantities of two species of cotton at 7d. and 6d. a pound, and one-third in money; how much money, and how many pounds of each sort of cotton did he receive?

6. A grocer mixed 106lbs. of tea at 3s. 4d. a pound, 75lbs. at 5s. 2d., and 94lbs. at 5s. 5d., and sold the mixture at 5s. a pound; what was the gain?

7. Of three sorts of barley a mixture is made, namely, 8 bushels at 3s. 6d., 15 at 3s. 8d., and 28 at 3s. 10d.; what is the value of a bushel of the mixture?

IX.

1. Find the yearly average of the profits or losses in seven years of a trader who gained in the first, third, fourth, and sixth years the

sums £520, £200, £700, and £1,000; and lost in the second, fifth, and seventh years £100, £250, and £320 respectively.

2. According to a Parliamentary return for the year 1872-3, the dividends on India Stock charged with income tax was £7,032,247; on Colonial Stocks, £2,839,776; and on Foreign Stocks, £9,341,193; find the income tax on each of these dividends, and the whole amount at the rate of fourpence in the pound.

3. The rental of a parish is £19,380; how much in the pound will produce a rate of £403 15s.?

4. A farmer rents a farm of 500 acres, on the following terms: he pays a fixed rent of 15 shillings an acre, and a corn rent of 120 quarters of wheat, 80 quarters of barley, and 60 quarters of oats. The prices of wheat, barley, and oats being respectively 48s. 6d., 25s., and 19s. 6d. per quarter; find his rent per acre.

5. In a certain parish the tithe rent-charge, valued at £740, was commuted for equal portions of wheat, barley, and oats, whose prices were 6s. 8d., 3s. 2d., and 2s. 6d. a bushel respectively. When barley and oats were at 2s. 9d. and 2s. respectively, what must be the price of wheat, that the tithe rent-charge may be worth £780?

6. In 1834, on the abolition of slavery in the West Indies, 20 millions sterling was voted by the British Parliament to compensate the slave-owners, whose slaves were valued at £45,282,000. What was the average compensation the slave-owners received for each slave, supposing each slave of same value?

X.

1. Divide £1000 among three persons, so that the first may have 3 times as much as the second, and the third as much as the first and second together.

2. The sum of £984 is to be divided between 4 men, *A*, *B*, *C*, *D*, in such a manner, that for every £3 given to *A*, *B* is to receive £5, *C* £7, and *D* £9; what sum did each receive?

3. Divide £1000 among four persons *A*, *B*, *C*, *D*, so that *C* shall have £5 more than *D*, *B* £5 more than *C*, and *A* £5 more than *B*.

4. The sum of £72 is to be divided among 24 men, 36 women, and 72 children, so that the shares of two men shall be equal to that of three women, and each woman's share equal to the shares of two children; what will be the share of each?

5. The sum of £204 6s. 3d. is to be divided between 6 men, 3 women, and 12 children: for every pound each woman receives, each man is to receive two pounds, and each child ten shillings; how much will they severally receive?

6. The sum of £20 is to be divided among 42 persons, in such a manner that 25 of them shall have 7s. 6d. each, and the rest shall share equally in the remainder; find the share of each of the latter.

7. In a manufactory there are employed 3 foremen at 3s. 4d. a day,

54 workmen at 2s. 3d., and 21 boys at 1s. 6d. They work 6 days in the week. How much has the master to pay in wages per week, and how much per year?

8. A manufacturer employs 50 men and 35 boys, who respectively work 12 and 8 hours per day during 5 days of the week, and half time on the remaining day. Each man receives 6d., and each boy 2d. per hour. What is the amount of wages paid in the year?

9. There is in a manufactory a certain number of workmen who receive 50s. a week, twice as many who receive 31s. 6d. a week, and eleven times as many who receive 14s. a week, and the total amount of the workmen's wages for one week is £93 9s.; find the number of workmen.

XI.

1. Distinguish between abstract and concrete numbers, and show that the distinction is necessary to avoid error in the interpretation of the results of numerical calculations.

2. What is the standard lineal measure of the United Kingdom? Give a brief account of its history, where is it preserved, and how is it kept correct?

3. How many inches are there in 24 lineal yards 2 feet and 11 inches? Conversely, how many yards, feet, and inches are there in 899 lineal inches?

4. How many yards, feet, and inches are there in a million lineal inches? Reverse the process.

5. Find how many inches there are in 5 miles, 1 furlong, 151 yards.

6. Divide 10,841 lineal yards by 37 and by 37 lineal yards, and state the nature of the quotients.

XII.

1. A carriage wheel revolves 3 times in going 11 yards; how many times will it revolve in going two miles and a half?

2. The circumference of the fore wheel of a carriage is 8 feet, and that of the hind wheel is 10 feet; in what distance will the fore wheel make 100 revolutions more than the hind wheel?

3. The larger wheel of a carriage being 24 inches in circumference longer than the smaller, makes 440 revolutions in a mile; how many will the smaller make in a mile?

4. There are four places on the same road in the order A, B, C, D. From A to D the distance is 1,463 miles; from A to C, 728 miles; and from B to D 1,317 miles. How far is it from A to B, from B to C, and from C to D?

5. From Ephesus to Cunaxa, Xenophon with the army of Cyrus marched 16,050 stadia of 202 yards 9 inches each in 93 days. Find the average length of a day's march in miles and yards.

6. A pound of cotton may be spun into a thread 134,000 yards

long; what weight of this thread would reach round the earth, a distance of 25,000 miles?

7. If a yard contain 36 inches or 12 palms or 4 spans or 2 cubits, express a mile in palms, spans, cubits, respectively.

8. Three French kilomètres are as much under two English miles as five kilomètres are above three miles; find how many yards make a kilomètre.

XIII.

1. If the magnitude of the lineal unit be given, what are the corresponding units of area and volume? Exemplify, when the lineal unit is 12 inches.

2. Reduce 6,698,411 square inches to square yards, feet, and inches. Reverse the process.

3. Reduce 49 acres, 28 poles, 10 yards, 112 inches to inches.

4. How many square yards are there in 139,968 square inches?

5. Divide a square foot by 18 lineal inches.

6. How many square inches are there in the area of a rectangular parallelogram whose adjacent sides are respectively 3 feet 4 inches and 5 feet 7 inches?

XIV.

1. A rectangular court, the sides of which are 300 feet and 200 feet, has a walk 20 feet wide running round it; what is the area of the walk?

2. Find the area of a field in acres and yards whose sides are respectively 50 chains 40 links and 50 chains 25 links.

3. How many acres of land would be required for a road 50 miles long and 22 yards broad?

4. If the hide of land mentioned in Domesday Book contained 4 plow lands, and one plow land 4 yard lands, and the yard land 30 acres, find how many acres were estimated to the hide of land.

5. How many allotments can be made from 25 acres of land, allowing 1000 square yards to each allotment?

6. Find the number of men required for forming a hollow square four deep, having 100 men on each side.

7. If a halfpenny piece be one inch in diameter how many can be laid in rows touching each other on a table which is 7 feet 6 inches long and 3 feet 4 inches wide; and what is their amount?

8. A labourer engaged to reap a field of corn for 5s. an acre, but leaving 6 acres not reaped, he received £2 10s.; of how many acres did the field consist?

9. What would be the cost of painting the walls and ceiling of a room whose height is 10 feet, breadth 15 feet, and length 24 feet, at 6d. a square foot?

10. If eight strips of carpet 39 inches wide cover a floor, how wide must a carpet be in order that 6 strips may cover the floor?

11. If 68 bales of linen contain 67,048 yards, and each bale contains 34 pieces, and each piece the same number of yards, how many yards are there in each piece?

XV.

1. What is meant by saying that the solid content of a cube is the cube of one of its edges, and of the area of a square the square of one of its sides?

2. How many cubic yards, feet, and inches are there in 496321675 cubic inches? Reverse the process.

3. From a beam 8 inches thick and 9 inches wide, how many feet must be cut off to contain a cubic foot?

4. Multiply 10 square inches by 10 lineal inches, and state the nature of the product.

5. Divide 2 cubic feet 108 inches by 162, and by 1 square foot 18 inches, and state the nature of the quotients.

6. How many bricks, 9 inches long and 4 inches wide, will be required to pave a courtyard 360 feet long and 220 feet wide?

7. What is the content in cubic feet of a bed of coal $5\frac{1}{2}$ miles long, 3 broad, and 30 feet deep?

8. A room contains 24,000 cubic feet of space, its length is 60 feet, and its breadth 40 feet; what is its height?

9. How many bricks are there in a wall which is 120 yards long, 15 feet high, and 24 inches thick, supposing a brick to be 9 inches long, 4 wide, and 3 thick?

10. How many leaden bullets 1 inch in diameter can be packed in a box whose internal length is 2 feet 4 inches, and depth and breadth each 18 inches?

XVI.

1. If 277280 cubic inches of water weigh 10000 pounds Avoirdupois, how many will weigh 1000 ounces?

2. If a cubic foot of water weigh 1000 ounces, how many hundred weight of water will a full cistern contain, whose length is 16 feet, breadth 10 feet, and depth 7 feet?

3. What must the depth of a cistern whose length and breadth are 24 feet and 8 feet, which shall contain as many cubic feet as another whose length, breadth, and depth are respectively 12 feet?

4. If a cistern be 12 feet long, 12 feet wide, and 12 feet deep, how many cubic feet does it contain, and how many square feet are there in its internal surface?

5. What must be the depth of a cistern 6 feet long and 4 feet wide which shall contain as many gallons as are contained in two cisterns, one containing 100 gallons and the other 200 gallons, if a gallon of water weigh 10 pounds and a cubic foot of water weigh 1000 ounces?

6. If the pressure of the atmosphere at the surface of the earth, when the barometer stands at 30 inches, be about 15 pounds on the

square inch, what is the pressure in pounds on the surface of the human body, supposing it to be 15 square feet? What would be the difference of the pressure when the barometer stands at 29 inches?

7. A wine merchant bought 5 pipes of wine for £138 12s., and one pipe became damaged; at what rate per gallon must he sell the remaining 4 pipes, so as not to be a loser by the purchase?

XVII.

1. The sum of the diameters of the sun and the earth being 894056 miles, and the diameter of the sun being 112 times that of the earth, find the diameter of the earth.

2. Sir John W. Herschel reckoned that the length of the earth's polar diameter is 41,707,796 feet, which is within one furlong of five hundred and a half million inches. What is the exact difference?

3. The volume of the sun is reckoned about 1384500 times that of the earth, and the volume of the earth 80 times that of the moon; how many times does the volume of the sun contain that of the moon?

4. The distance of Mercury from the sun is 5917938 French mètres, and of Uranus 291720130; how many times Mercury's distance is the distance of Uranus from the sun?

5. If the planet Saturn be 908,723,000 miles distant from the sun, and the earth 91,713,000 miles, shew that Saturn is more than 9 times, but less than 10 times, the earth's distance from the sun.

6. The distance of the star a Centauri from the earth is reckoned to be about nineteen billion of miles, and the distance of the sun is about 91,713,000 miles; by how much is the star more distant than the sun?

XVIII.

1. How many grains of gold are contained in 10lb. 11oz. 12dwts. 13grs.? Prove the truth of the result by the reverse process.

2. What is the weight of 125 bars of silver, of which there are 20, each weighing 3lb. 5oz. 7dwts. 20grs.; 67, each weighing 7lb. 3oz. 15dwts. 15grs.; and each of the remaining bars weighs 4lb. 11oz. 17dwts. 17grs.?

3. If one pound Troy weight of Standard silver be coined into 66 shillings, how much silver would be required to coin a million shillings?

4. At the English mint 40 pounds of Standard gold are coined into 1869 sovereigns. What is the mint price per ounce Troy of Standard gold?

5. In the purchase and sale of bullion, bars of silver from 50 to 80 pounds Troy have an allowance over of 10 pennyweights, and ingots of gold from 15 to 20 pounds have 6 grains allowed; what are the amounts of the allowances on the purchase of 1000 pounds of gold, and on 1000lbs. of silver?

6. In Exod. xxxviii. 25, 26, it is stated that 100 talents 1775 shekels of silver were collected from 603550 people for the work of the tabernacle, and that each person contributed half a shekel; determine the number of shekels in the talent.

XIX.

1. If 9 grains of silver are worth five farthings, what should a crown weigh?

2. If an ounce of gold be worth £3 17s. 10½d., what is the value of one grain?

3. If 1lb. 6oz. 2dwts. 8grs. of gold be worth £68 2s. 6d., how much is that the ounce?

4. What is the price of a silver cup weighing 1lb. 10oz. 12dwts. 12grs., at 6s. 6d. an ounce?

5. If a pound of silver cost £3 6s., what is the price of a salver which weighs 7lb. 7oz. 10dwts., subject to a duty of 1s. 6d. per ounce, and an additional charge of 1s. 10d. per ounce for the workmanship?

6. If a talent of silver be worth £357 11s. 10d., what is the value of a shekel, if 3000 shekels make a talent?

7. The gold procured from Australia in 6 months in 1851 amounted to 209,096 ounces. In 1861 the New Zealand gold-fields yielded 228,292 ounces in the same time. What is the excess in weight and value, at £3 17s. 10½d. per ounce, of the average monthly return from New Zealand over that from Australia?

XX.

1. What is the aggregate of the following weights: 2 tons, 100lbs., 75 cwt., 14lbs., 5drs., 2000lbs., 5000 ounces, and 20,000 drams?

2. Subtract 25cwt. 2qrs. 5lb. 5drs. from 5 tons 7cwt. 1qr. 20lbs. 3drs.

3. Multiply 12 tons 7cwt. 15lb. 7oz. by 25, and reverse the operation.

4. How many tons, hundredweights, &c., are there in a million drams Avoirdupois?

5. Reduce 9 tons 2cwt. 3qrs. to drams, and reverse the operation.

6. Divide 512cwt. 3qrs. 7lb. by 14cwt. 73lbs.

XXI.

1. Shew that in £59 12s. 1¾d. there are the same number of farthings as there are pounds in 25 tons 10cwt. 3qrs. 19lb.

2. In the year 1787 a new copper coinage, made by Mr. Boulton at Birmingham, was issued from the Mint for circulation. It consisted of pieces of four different sizes and values. The twopenny pieces amounted in nominal value to £6019 15s. 8d.; the penny pieces to £183,177 18s. 6d.; the halfpenny pieces to £88,506 18s. 4d.; and the farthing pieces to £4370 13s. 2d.; find the total number of pieces in this coinage.

3. A loaded waggon weighs 2 tons 3cwt. 2qrs. 23lb.; the waggon by itself weighs 18cwt. 3qrs; the load consists of 215 packages, each of the same weight; find the weight of each.

4. How many packets of 11lb., 13lb., 17lb., and of each the same number, may be made out of 3 tons 2cwt. 1qr. 23lbs.?

5. If one pound of copper be coined into 24 penny pieces, how many may be coined from a ton of copper, and what is their value?

XXII.

1. By how many grains does an ounce Troy differ from an ounce Avoirdupois?

2. Reduce 281750 grains Troy to pounds Troy, and find the equivalent number of pounds Avoirdupois.

3. From one ton Avoirdupois of the lead ore of a certain mine, 1lb. 11oz. 6dwts. 16grs. Troy of silver is extracted; find how much silver can be extracted from 17 tons 14cwt. 2qrs. of the same ore.

4. If 1lb. 8oz. Avoirdupois is half the weight of 3lb. 7oz. 15dwts. Troy, find the number of grains in one pound Avoirdupois.

5. A person purchases goods at the rate of six shillings per pound Troy weight, and sells them again by Avoirdupois weight; at what rate must he sell per ounce, so as exactly to reimburse himself?

XXIII.

1. Explain clearly what is meant by an unit of angular measurement. What is the circular measure of an angle, and upon what geometrical property does this measure depend?

2. The angle subtended at the centre of a circle by a chord equal to the radius is constant for the same circle. How many French degrees and how many English degrees are contained in this angle?

3. If 90 English degrees are equal to 100 French degrees, and one English degree contain 60 minutes and one French degree 100 minutes, give a rule for changing French minutes into English minutes, and English degrees into French degrees, and conversely.

4. What number of degrees, minutes, &c., in the English scale corresponds to 30° 15′ 25″ in the French scale of angular measure?

5. What is the difference of latitude between St. Peter's at Rome, 41° 53′ 54″ N., and St. Paul's in London, 51° 30′ 49″ N.?

6. What is the difference of latitude between Greenwich Observatory, which is 51° 28′ 38″ North, and that at the Cape of Good Hope, which is 33° 36′ 3″ South?

7. What is the mean of the following observed altitudes of a star, 50° 17′ 35″, 50° 13′ 12″, 50° 15′ 10″, and 50° 20′ 50″?

8. What is the difference of longitude between Athens and Rome, and between Rome and Philadelphia in the United States; if the longitude of Athens be 23° 47′ 0″ East; the longitude of Rome, 12° 27′ 14″ East; and the longitude of Philadelphia, 75° 10′ 0″ West, from Greenwich?

XXIV.

1. Reduce 12 weeks, 6 days, 5 hours, 10 min., 12 sec., to seconds, and reverse the operation.

2. How many days, hours, &c., are in a million seconds?

3. Find the number of seconds from 5 o'clock on Monday afternoon to 7 on Wednesday morning.

4. In one tropical year there are 365 days 5 hours 48min. 51secs.; how many seconds are there in 16 years 122 days?

5. How many hours have elapsed since the birth of Christ to the year 1849, supposing each year to consist of 365 days 6 hours?

6. In a Julian year seven of the months have thirty-one days each, four have thirty each, and one has twenty-eight or twenty-nine. How many minutes are in an ordinary year and in a leap-year?

7. How many seconds are contained in the lunar month of 29 days 12 hours 44 minutes 3 seconds?

XXV.

1. Multiply 2 years, 9 months, 6 days, by 31, and reverse the process.

2. A sidereal day is less than a solar day by 3 minutes, 56 seconds; in how many days will the difference amount to 24 hours?

3. Find the average of the following times, reckoned from noon: 2hrs. 12min. 28sec. p.m.; 10hr. 31min. 14sec. a.m.; 3hrs. 15min. 44sec. p.m.; 1hr. 36min. 35sec. p.m.; 11hrs. 40min. 39sec. a.m.; 4hrs. p.m.; and 1hr. 23min. 15sec. p.m.

4. It was high-water at London Bridge on seven successive days at 9h. 17m. a.m., 10h. 33m. a.m., 11h. 37m. a.m., noon, 0h. 52m. p.m., 1h. 38m. p.m., 2h. 24m. p.m.; what was the average time during the week?

XXVI.

1. In reckoning years from the epoch of the Christian era, state a rule for finding what years of this era are leap-years in different centuries. Are the years 1784 and 1874 leap-years?

2. How is it known when ordinary years begin and end with the same day of the week? What is the difference in leap-years?

3. Shew from the fifth and seventh chapters of the Book of Genesis that the Deluge happened in the 1656th year from the creation of the world, according to the Hebrew account.

4. How long did the Roman empire endure from its foundation till the year A.D. 800, when Charlemagne was crowned Emperor of the West by Pope Leo III, if we admit that Rome was a regal state, governed by 7 kings during 244 years; a republic under consuls until the time of Augustus, a period of 446 years; an empire under 57 emperors, during 519 years, from the time of Augustus to Augustulus, who was deposed by Odoacer, King of the Heruli; under this last and

8 kings of the Ostrogoths, during 92 years; and lastly under 22 Lombardian kings during 206 years?

XXVII.

1. How long would it require to count 800 millions of money, at the rate of £100 a minute, supposing the accountant to be employed six days in the week and ten hours daily?

2. If 4 artillerymen can fire a gun 48 times, and 5 men 54 times in an hour, how much longer time will be required for firing 32448 shots from 26 guns, when there are 4 men to each gun, and when there are 5 men?

3. If 6 companies of 54 soldiers each fire 6804 shots in 3 hours, how many shots will 8 companies of 81 men each fire at the same rate in 2 hours?

4. A regiment contains 1000 men, of which 50 are officers, how many men are there to each officer?

5. The population of the county of Middlesex increased from 1886576 in 1851 to 2206485 in 1861; what was the average yearly rate per cent. of increase in the ten years?

6. An article which costs 12s. 6d. will last 6 months, and another which costs 15s. will last 9 months; which is the cheaper of the two? and how much in a year?

XXVIII.

1. If a clock gain 12 minutes in a day, what is the gain in one minute?

2. How many times will a pendulum vibrate in 24 hours that makes 5 vibrations in 2 seconds?

3. Express 59° 27′ 30″ of longitude, in hours, minutes, and seconds of time; and conversely.

4. On 21 June the sun rises 3hrs. 45m. a.m., and sets 8hrs. 18min. p.m.; and on 21 December the sun rises at 8hrs. 6min. a.m., and sets 3hrs. 51 min. p.m. in the latitude of London; find the difference in the lengths of the 21st June and 21st December.

5. The transit of the planet Venus across the sun's disk happened on the morning of 9th December, 1874; the first contact at ingress began 1hr. 45min. 58sec. a.m., Greenwich mean time, and the last contact at egress ended at 6hrs. 26min. 54sec. a.m. How long was the planet in crossing over the sun's disk?

XXIX.

1. In how many weeks can a journey of 12600 miles be completed by travelling six days each week at the rate three miles an hour and seven hours each day?

2. If a man travel 300 miles in 10 days when the day is 12 hours long, in how many days can he travel 800 miles when the day is 16 hours long?

3. If a train 100 yards long pass a fixed object in 5 seconds, at what rate per hour is the train moving?

4. A fly-wheel revolves once every twenty seconds; how many revolutions does it make in twelve hours?

5. The length of the steps taken by two men are 3 feet and 4 feet, and the former takes five steps while the latter takes four. If both started from the same place to walk along the same road, how far has each walked when one is a hundred feet beyond the other?

6. How long will a column of 10,000 men, marching 4 deep, require to pass through a defile of 5 miles at the rate of 75 paces of 2 feet each in one minute, and the column on its march to extend 7500 feet?

7. If sound travel with a velocity of 1100 feet in one second, and the report of a gun is heard 15 seconds after the appearance of the flash, how far distant is the observer?

XXX.

1. If the length of the elliptic orbit of the earth round the sun be 596440000 miles, what is the average hourly and daily motion of the earth, supposing the earth makes one complete revolution in 365¼ days?

2. The mean distance of the sun from the earth is about 91,713,000 miles; what is the velocity of light, if the time in which the sun's light reaches the earth be 8min. 18sec.?

3. What is the difference in the hourly rates of the motion of two places, caused by the diurnal rotation of the earth on its axis, one on the equator and the other on the parallel of the latitude of London, supposing the circumference of these circles to be 24,900 and 15,120 English miles respectively?

4. How many seconds are there in a year of the planet Saturn, which consists of 10746 days 16min. 15sec. of time, as reckoned at the earth?

5. How many years of the earth are equivalent to one year of Uranus, one year of the earth being 365 days 5hrs. 48min. 51sec., and of Uranus 30688 days 17hrs. 6min. 16sec.?

6. How many of Jupiter's years have elapsed since the birth of Christ to A.D. 1863, if one year of the earth be 365 days 5hrs. 48min. 51sec., and one year of Jupiter be 4330 days 4hrs. 39min. 2sec.?

7. The periodic time, or length of the year of the planet Uranus, is 84 years 5 days 19hrs. 41min. 36sec., and of the earth 365 days 5hrs. 48min. 57sec. Shew that one year of Uranus is greater than 19, but less than 20, years of the earth.

8. If Sirius, one of the brightest of the fixed stars, which is probably 592,200 times farther from the earth than the sun, were suddenly extinguished, for how long would it still appear to shine to the

inhabitants of the earth, supposing, according to our present knowledge, the most probable value of the sun's mean distance from the earth to be 91,713,000 miles, and that light from the sun reaches the earth in 8min. 18sec.?

XXXI.

1. Shew that, in the year 1850, 7 times the number of days preceding June 18 were equal to 6 times the number following that date.

2. How many days were there from 10 January to 4 May, 1872, both days included?

3. If a father died in 1873 at the age of 96, and his son died in 1860 aged 50, how old was the father when the son was born?

4. If on Dec. 25, 1870, a man's age is 30000 days, find the day and year of his birth.

5. In the year 1873 a man was 75 years old, and 25 years ago he was twice as old as his son; what was his son's age in 1874?

6. If A be 10 years younger than B, and 5 years older than C in the year 1860; if C were 25 years old 15 years ago, when will B be 60 years old?

7. If the Easter term at Cambridge commences on a Wednesday and ends on a Friday, shew that it divides either on a Thursday at noon, or on a Sunday at midnight. In 1855, the Easter term began on Wednesday, 18 April, and ended on Friday, 6 July.

8. In the year 1860, the first day of January was a Sunday; find the next two years in which this will happen again.

9. There was a full moon on June 26, 1858, at 9hrs. 13min. a.m. The interval between successive full moons has since been on the average 29d. 12hrs. 47min. 30sec.; how many full moons happened until Dec. 31, 1873, and when did the last take place within that period?

10. The Times newspaper of Monday, 14 February, 1876, bears the number 28551. Supposing the paper to have been published every week day without intermission and numbered consecutively, give the day of the week, month, and year when No. 1 was published.

RESULTS, HINTS, ETC., FOR THE EXERCISES.

I.

2. 69 fourpenny pieces. 3. 1680 half-crowns, 3360 sixpences, 5040 fourpenny pieces. 4. 133333 half-crowns and 10 pence over. 31746 half-guineas and 4 pence over. 5. 5530 farthings. 7. When the given number of farthings is an exact number of pounds.

II.

1. £1395 19s. 7½d. 3. £2095891 11s. and 29316402 pieces of coin. 4. *A* owes to *B* £122 1s. 2¾d. 5. Increase of 1866 over 1865, £13,464,865 : Decrease of 1867 under 1866, £10,006,489.

III.

1. The products are respectively £23 3s. 11½d.: £33 15s. 11¼d.: £45 7s. 11d.: £56 19s. 10¾d.: £67 11s. 10½d.: £80 3s. 10¼d.: £92 15s. 10d.: £101 7s. 9¾d. 2. £521 7s. 9¾d.: £551 3s. 8¼d.: and £2606 19s. 1¾d. 3. Multiply the sum by 1000 and subtract the given sum from the product. 4. See Art. 4, note. 5. Art. 7, note. 6. The quotient is 6s. 11¼d. and £12 14s. 3¾d. remainder. 7. 5d. and 5 times. 8. 328 times. 9. See Art. 4, note. 10. See Section II. pp. 47, 48, and Section VII. p. 3.

IV.

1. £24 1s. 2. 19 persons. 3. 123 persons and 2s. 6¾d. over. 4. If the number of articles divided by 40 be subtracted from 3 times the number, the difference will be the price required. 5. 6s. 8d. is one-third of £1. 6. If £3 16s. 6d. be divided by 6, the quotient gives the value of one article, and if this be multiplied by 13 the value of 13 is known. 7. Divide the sum by 365. 8. £8 5s. 3d. shared each, and 4s. 9d. over.

V.

1. 420 crowns. 2. 605 francs. 3. 61009 coins. 4. 6680 ducats. 5. 592 half-crowns. 6. 31941 roubles. 7. 1 rupee is worth 2s. 3d. 8. £211 13s. 4d.

VI.

1. 56 half-crowns, 140 shillings, and 280 sixpences. 2. 114 crowns and 114 shillings. 3. 22 penny, 44 shilling, and 66 crown pieces. 4. *A* had £200 and *B* 200s. 5. The first has £20, the second £100, and the third £400. 6. 20 of each. 7. 5d., 30s., and £210. 8. 60 sixpenny pieces, 30s., and 12 half-crown pieces.

VII.

2. 5d. in the pound. 3. £28 1s. 8d., the difference. 4. £1178 2s. 6d. 5. £131 2s. 6d. 6. 14s. 5½d. daily expenditure, and 2s. 8½d. over.

7. Before the strike: 52 weeks' wages at 15s. a week... ... ... = 780s.
Savings at end of the year ... ... = 26s.

52 weeks' expenses of living ... ... = 754s.

After the strike: Wages increased 2s., 52 weeks at 17s.... ... = 884s.
Expenses of living, &c., increased by 3d. in the shilling, gives an addition of 188s. 6d. to 754s., and yearly expenses ... ... = 942s. 6d.

Instead of additional saving, at end of year he is in debt... ... ... 58s. 6d.

8. £698 2s. 6d. in debt. 9. £240 18s. saved in 3 years. 10. £1213 10s. 10d. clear income.

VIII.

2. £17 12s. 3d. the balance in favour of sugar. 3. 66lb. of tea. 4. 6d. lost on the sale. 5. £21 13s. 4d. in money, and 800lb. of each species of cotton. 6. £6 4s. 4d. gain. 7. 3s. 8¾d. value of a bushel, and 1¾d. over.

IX.

1. £250 average yearly gain in 7 years. In the transactions of commerce it is frequently necessary to find the average price of several sets of units which may have been bought or sold at different times and at different prices. And it may be added that the word average is not restricted to matters of commerce, but is applied to almost all human events and transactions, as the finding for any given number of years the births and deaths in a city, a county, or a nation.

2. Income tax on India Stocks £117204 2s. 4d.: on Colonial Stocks £47329 12s.: on Foreign Stocks £155686 11s.: the whole amount £320220 5s. 4d. 3. 5d. in the pound. 4. £1 13s. an acre. 5. The price of wheat must be 8s. 3d. a bushel. 6. Average compensation 8s. 10d. for each slave, and 9s. over.

X.

1. If the second have 1 share, then the first has 3 and the third has 4, and the sum of all the shares is 8 shares, and the value £1000; they are £125, £375, and £500. 2. *A's* share £123, *B's* £205, *C's* £287, *D's* £369. 3. It appears that *C* has £5 more than *D*, *B* £10 more than *D*, and *A* £15 more than *D*, and the four shares are equal £30 and four times the share of *D*. But the four shares are £1000; hence £30 and 4 times *D's* share is equal to £1000, and 4 times *D's* share is £970, and *D's* share is £242 10s., *C's* share £247 10s., *B's* share £252 10s., and *A's* share £257 10s. 4. Share of a child 6s. 8d., of a woman 13s. 4d., of a man 20s. 5. A man £19 9s. 2d., a woman £9 14s. 7d., a child £4 17s. 3½d. 6. 12s. 6d. each. 7. £48 18s. weekly, and 52 times that sum in the year. 8. In the year are 52 half-working days and 261 whole days; the total amount of wages for the year is £3974 13s. 4d. 9. 7 workmen at 50s. a week, 14 at 31s. 6d. a week, and 77 at 14 a week.

XI.

1. See Art. 2. 2. See Section III. p. 16. 5. 330156 inches. 6. 293 lineal yards; 293 an abstract number, denoting the number of times 37 lineal yards are contained in 10841 yards.

XII.

1. 1200 revolutions. 2. The hind wheel makes 400 revolutions while the fore wheel makes 500 in going over 4000 feet. 3. 528 revolutions. 4. *A* to *B*, 146 miles: *B* to *C*, 582 miles: and *C* to *D*, 735 miles. 5. 19 miles 1464 yards daily, and 40½ yards over. 6. 328lb. 5oz. 11drs., with 47drs. over. 7. One mile contains 21120 palms, 7040 spans, or 3520 cubits. 8. 1100 yards.

XIII.

1. See Art. 6. 2. 5168 square yards 4ft. 107in. 3. 308470144 square inches. 4. 108 square yards. 5. 8 lineal inches. 6. 2680 square inches.

XIV.

1. 18400 square feet. 2. 253 acres 1258 yards. 3. 400 acres. 4. 480 acres. 5. 121 allotments. 6. 1536 men. 7. The number is 3600 halfpence, in value £7 10s. 8. The field consisted of 16 acres. 9. £28 10s. 10. 52 inches wide. 11 29 yards in each piece.

XV.

1. See Sec. VI. 2. 106378 cubic yards 24 feet 1035 inches. 3. 2 feet.

4. 100 cubic inches. 5. 22 cubic inches and 22 lineal inches. 6. 312800 bricks. 7. 2613600 cubic feet. 8. 10 feet. 9. 172800 bricks. 10. 8072 bullets.

XVI.

1. 1 cubic foot nearly. 2. 625 cwt. 3. 9 feet deep. 4. 1728 cubic feet and 720 square feet. 5. 4 feet deep. 6. 32400lb. 7. 5s. 6d. a gallon.

XVII.

1. 7912 miles. 2. The exact difference is 179yds. 0ft. 4in., which is in defect of a furlong. 3. 110760000 times. 4. More than 206 times the distance of the earth from the sun.

XVIII.

2. 748lb. 8oz. 16dwts. 11grs. 3. 15151lb. 6oz. 3dwts. 15grs. nearly. 4. £3 17s. 10½d. per ounce. 5. 6 grains on 20lb. of gold give 300 grains allowed on 1000lb.: and 10dwts. on 80lb. of silver give 125 grains allowed on 1000lb. 6. 3000 shekels in one talent in the time of Moses.

XIX.

1. 18dwts. 2. A little less than 2d. 3. £3 15s. 2½d. nearly. 4. £7 7s. very nearly. 5. £41 7s. 10d. 6. 2s. 6d. nearly. 7. Excess of weight 73196 ounces: in value £285064 7s.

XX.

1. 6 tons 17cwt. 1qr. 12lb. 4oz. 5drs. 2. 1cwt. 3qrs. 14lb. 14drs. 4. 1 ton 14cwt. 98lb. 4oz. 6. The quotient is 35, an abstract number denoting the number of times 14cwt. 73lb. is contained in 512cwt. 3qrs. 7lb.

XXI.

1. Express the money in farthings, the weight in pounds. 2. 721174 twopenny pieces: 46362702 penny pieces: 42483320 halfpenny pieces and 4195832 farthing pieces: the total number 93763028. 3. Weight of each packet 13lb. 4. 171 packages. 5. 53760 penny pieces, value £224.

XXII.

1. The ounce Troy exceeds the ounce Avoirdupois by 42 grains and a half. 2. 48lb. Troy and 5720 grains over: 40lbs. Avoirdupois and 1750 grains over. 3. 413oz. 11dwts. 16grs. Troy of silver. 4. 7000grs. in 1lb. Avoirdupois. 5. At 11d. per ounce Avoirdupois very nearly.

XXIII.

1. See Art. 8. 2. 30 English degrees, but 33 and one-third French degrees. 3. Since 90 English degrees are equal to 100 French degrees, 1 English degree is equal to $\frac{10}{9}$ or $1+\frac{1}{9}$ French degree: and 1 French degree is equal to $\frac{9}{10}$ or $1-\frac{1}{10}$ English degree. Hence, any number of English degrees increased by one-ninth are changed to French degrees; and any number of French degrees diminished by one-tenth are changed into English degrees. 4. 33° 30′ 10″ English. 5. 9° 36′ 55″. 6. 85° 4′ 41″. 7. 50° 16′ 42″ nearly. 8. Difference of longitude between Athens and Rome, 11° 19′ 46″: difference of longitude between Rome and Philadelphia, 87° 37′ 14″.

XXIV.

1. 7794612 seconds. 2. 11 days 3hrs. 46min. 40sec. 3. 223200 seconds. 4. 515451696 seconds. 5. 16208234 hours. 7. 2551443 seconds.

XXV.

2. Between 46 and 47 days. 3. 8hrs. 56min. 30sec. a.m. 4. 8hrs. 51min. a.m.

XXVI

1. See Section IV. pp. 17, 18. 2. Section IV. p. 17. 3. Take the years between the births of the eldest sons. 4. 1507 years.

XXVII.

1. 2277 weeks 4 days 6hrs. 20min. 2. 1hr. 53min. 20sec. 3. 9072 shots. 4. 19min. 5. The average yearly increase per cent. for each of the 10 years was 3199. 6. That which costs 15s. is the cheaper by 5s. in a year.

XXVIII.

1. Half a second. 2. 216000 vibrations. 3. As 15° of longitude correspond to 1 hour of time, the equivalent in time is 3hrs. 57min. 50sec. 4. 8hrs. 48min. 5. 4hrs. 40min. 56sec.

XXIX.

1. 100 weeks. 2. 20 days. 3. Between 40 and 41 miles an hour. 4. 2160 revolutions. 5. One has walked 1600 feet, and the other 1500 feet. 6. 3hrs. 46min. 7. 3 miles 220 yds.

XXX.

1. Above 68040 miles an hour. 2. Light moves more than 184000 miles in one second of time. 3. 1034 miles an hour at the equator, and 630 miles an hour in the latitude of London. 4. 928455375 seconds. 5. Saturn's year is between 84 and 85 years of the earth. 6. More than 157, but less than 158 years of Jupiter. 8. Between 9 and 10 years.

XXXI.

1. The question implies that the 18th June is to be omitted: there are 168 days before, and 196 days after that day. 2. 116 days. 3. 33 years. 4. Section IV. pp. 17, 18. 5. 26 years. 6. In 1845 *C* was 25 years of age, *A* 30, and *B* 40. In 1865 *B* will be 60 years of age. 8. See Section IV. pp. 17, 18. 9. Find the time between Jan. 26, 1858, 9hrs. 13min. a.m., and the same point of time a.m. on Dec. 31, 1873. Ascertain how many intervals occur in that period, reckoning the first full moon from June 26th, 1858.

10. See Section IV. pp. 17, 18.

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SECTION VIII.

MEASURES AND MULTIPLES.

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ON PRIME AND COMPOSITE NUMBERS.

MEASURES AND MULTIPLES.

ART. 1. There are certain relations and properties of numbers of extensive use in almost all arithmetical operations, which are also necessary in the arithmetic of fractions and decimals.

In the series of natural numbers, 1, 2, 3, 4, 5, 6, &c., a distinction may be observed of odd and even numbers. An odd number is one which cannot be divided into two equal whole numbers, as 1, 3, 5, 7, 9, &c. An even number is one which can be divided into two equal whole numbers, as 2, 4, 6, 8, &c.

There is another, a more important division of the natural numbers into two classes, one class consisting of numbers, each of which is divisible only by 1 and a number equal to itself, as 2, 3, 5, 7, 11, &c.; and the other class consisting of numbers which admit of other divisors, as 4, 6, 8, 9, 10, &c. The numbers in the former class are called *prime numbers;* and those in the latter class, *composite numbers*.

DEF. A prime number is one which can be exactly divided only by unity and a number equal to itself.

DEF. A composite number is a number which is composed of the product of two or more prime numbers; or, a composite number may be defined to be a number which admits of other divisors than unity and a number equal to itself.

DEF. A *multiple* of a number is any number of times that number; as 12 is a multiple of 3 and of 4, for 12 is 4 times 3, and 3 times 4. Both 3 and 4 are called *submultiples* of 12.

DEF. One number is said to be a *measure* of another, when the former is contained in the latter a certain number of times exactly; as 3 is a measure of 12, for 3 is contained 4 times exactly in 12.

The terms *multiple* and *measure* are only other names for dividend and divisor in cases where there is no remainder after division, and are thus related: since 3 is a divisor of 12, it follows that 12 is a multiple of 3.

DEF. A *common measure* or *common divisor* of two or more numbers is a number which will divide exactly each of them; and the greatest number which will exactly divide each of them is called their greatest common measure; thus 3 is a common measure of 12 and 18, but 6 is their greatest common measure.

When two or more numbers have no common measure except 1, they are said to be prime to each other, as 2 and 3, 4 and 5, 6 and 35; and all *prime numbers* are prime to one another; but *composite numbers* may be, or may not be, prime to one another.

Every prime number is prime to all numbers which are not multiples of itself.

Two or more numbers are said to be *commensurable* when any integral number will measure or divide each of them exactly; and if no integral number can be found which will exactly measure each of them, they are said to be *incommensurable.* Unity being a measure of every number, is not considered as a measure.

Def. A number which is produced by the multiplication of two or more numbers is called a *common multiple* of each of them; as 12 is 3 times 4, and 4 times 3; 12 is a multiple of 3 and a multiple of 4, and therefore a common multiple of 3 and 4.

Def. The *least common multiple* of two or more numbers is that number which is the least multiple of each of the given numbers. Thus 12 is the least common multiple of 4 and 6; for 12 is 3 times 4, and 2 times 6; and there is no number less than 12 which contains 4 and 6 a less number of times.

Or. A number which is divisible by two or more numbers is also defined to be a common multiple of these numbers; and the least number which is so divisible is their least common multiple. Thus 12 is a common multiple of 2 and 3; for 12 is exactly divisible by 2 and by 3: but 12 is the least common multiple of 4 and 6.

If one number measure another, it will measure any multiple of that number; as if 3 measure 6, it will measure 2, 3, 4, 5, &c., times 6.

If one number be a common measure of two other numbers, it will measure their sum and their difference; as if 3 measure 12 and 18, it will also measure the sum 30 and the difference 6.

If one number measure two other numbers, it will measure the sum and difference of any multiples of them; as if 3 measure 12 and 18, it will measure the sum and difference of any multiples of 12 and 18, as 4 times 12 and 5 times 18; that is, 3 will measure 138 and 42.

2. Prop.—*To find when a number is divisible by* 9, *and by* 11, *numbers which are one less and one greater than the scale of notation.*

First, let the number 7236 be taken, and divided into parts which are, and which are not divisible by 9.

$$\begin{aligned}
\text{Then } 7236 &= 7000 + 200 + 30 + 6 \\
\text{and } 7000 &= 7 \times 1000 = 7 \times 999 + 7 \\
200 &= 2 \times 100 = 2 \times 99 + 2 \\
30 &= 3 \times 10 = 3 \times 9 + 3 \\
6 &= \qquad\qquad\qquad\quad 6 \\
\therefore\ 7236 &= (7 \times 999 + 2 \times 99 + 3 \times 9) + (7 + 2 + 3 + 6)
\end{aligned}$$

The number 7236 being divided into two parts, the first of which is obviously divisible by 9; and the second being the sum of the digits, is also divisible by 9; it follows that the whole number 7236 is divisible by 9 if the sum of its digits be divisible by 9. In the same way may be shewn that any number 7256 is not divisible by 9 when the sum of the digits is not so divisible.

As 9 is 3 times 3, it follows that a number is divisible by 3 if the sum of its digits be divisible by 3.[1]

Next, let the number 63547 be taken, and let it be divided into parts which are and which are not divisible by 11.

Here $63547 = 60000 + 3000 + 500 + 40 + 7$

$$
\begin{aligned}
\text{And } 60000 &= 6 \times 10000 = 6 \times (909 \times 11 + 1) = 6 \times 11 \times 909 + 6 \\
3000 &= 3 \times 1000 = 3 \times (90 \times 11 + 10) = 3 \times 11 \times 91 - 3 \\
500 &= 5 \times 100 = 5 \times (9 \times 11 + 1) = 5 \times 11 \times 9 + 5 \\
40 &= 4 \times 10 = 4 \times (11 - 1) = 4 \times 11 - 4 \\
7 &= 7
\end{aligned}
$$

$$
\text{Then } 63547 = (6 \times 11 \times 909 + 3 \times 11 \times 91 + 5 \times 11 \times 9 + 4 \times 11) + (6 - 3 + 5 - 4 + 7)
$$

The first part of which is obviously divisible by 11. Hence the number 63547 will be divisible by 11 if the second part $6 - 3 + 5 - 4 + 7$ be divisible by 11, that is if the difference between the sums of the digits in the odd and even places reckoned from the unit's place be divisible by 11.

In the same way may be shewn that a number is also divisible by 11, if the sum of the digits in the odd places be equal to the sum in the even places, or when the difference between the sums in the odd and even places is 0.

3. Prop.—*To find the prime numbers in order within any given limit.*

Let the natural numbers be written down in order, beginning with unity up to the given limit.

The number 2 being the only even prime number, all other even numbers being multiples of 2 are not prime numbers and may be struck out.

The number 5 being the only prime number ending with 5, all other numbers ending with 5 being multiples of 5 may be removed.

Also all numbers ending with one or more ciphers, being multiples of 2 and 5.

---

[1] The following simple properties are useful in resolving small composite numbers into their prime factors :—

A number is divisible by 10 when it ends with a cipher.

A number is divisible by 5 when it ends with 0 or 5.

A number is divisible by 2 when it ends with a cipher or an even number.

A number is divisible by 4 when the number composed of the first two digits reckoned from the unit's place is divisible by 4; and by 8 when the number composed of the first three digits is divisible by 8.

A number is divisible by 6 when it ends with an even number and is divisible by 3.

A number is divisible by 12 when it is divisible by 4 and by 3.

The criterion for discovering when a number is divisible by 7 is not of a simple nature.

Hence it follows that all prime numbers, except 2 and 5, will end either with 1, 3, 7, or 9.

Next, the number 3 being a prime number, every fourth number is divisible by 3, and may be omitted.

Similarly, the number 7 being a prime number, every eighth number is divisible by 7, and may be omitted; and so on, omitting all numbers divisible by each of the successive prime numbers in order, until at length all the composite numbers are sifted out. Then the remaining numbers will be the prime numbers contained in the given series.

As an exemplification of the process, let it be required to find all the prime numbers contained between the limits of 1 and 100.

Having written down in order the first 100 numbers; first, 2 and and 5 are prime numbers; omitting 2 and 5, and all multiples of them, there will be found 31 of the numbers left.

Next omitting 9, 21, 27, 33, 51, multiples of 3, a prime number; also 49, 77, 91, multiples of 7, a prime number.

And the remaining numbers are—

11 19 31 43 59 71 83
13 23 37 47 61 73 89
17 29 41 53 67 79 97

and of these no one of the larger numbers is a multiple of any one of the smaller prime numbers; it follows that these must be prime numbers.

And there are 25 or 26 (if unity be included) prime numbers in the first hundred of the natural numbers.[1]

---

[1] The sieve of Eratosthenes was the name given to a contrivance he invented for finding the prime numbers. His method consisted in writing the natural numbers in order, beginning with unity, and continuing the series to any extent. He then separated or sifted out all numbers that were not prime numbers, and by this means he ascertained the prime numbers in the order of their magnitudes. The problem of finding in general a prime number, beyond a certain limit, by a direct process, has engaged the attention of mathematicians, but no results of a satisfactory nature have yet been attained. No algebraical formula has yet been discovered which will contain prime numbers only, as all known formulæ for prime numbers fail, including other numbers besides prime numbers. The *κόσκινος*, or sieve, with some other fragments of the writings of Eratosthenes, was printed with the "Phænomena of Aratus" at Oxford in 1672.

Poetius, at the end of his Arithmetic, published at Leipsic in 1728, printed a table of the prime numbers and the factors of the composite numbers from 1 to 10,000. A similar table was published at Leyden by H. M. Anjema in 1767.

A larger table, extending from 1 to 101,000, was published at Halle by M. Kruger, at the end of his "Thoughts on Algebra." Lambert extended this table as far as 102,000, and reprinted it, with other tables, at Berlin, in 1770.

Peter Barlow, of the Royal Military Academy, published in 1814 a volume of Mathematical Tables. Table I. contains the prime numbers, and the factors of composite numbers from 1 to 10,000; also the reciprocals of these numbers calculated to ten places of decimals. It contains also the squares and cubes of these numbers with their square roots and cube roots each to seven places of decimals. Table V. is a register of all the prime numbers between 1 and 100,109.

4. **Prop.** *In the series of the natural numbers (at least for the first* 1000) *the prime numbers do not occur in any regular order of sequence in any classes into which the natural numbers can be divided, as may be seen from the following table, which contains the prime numbers in the first* 1000 *natural numbers:*[1]—

1
2
3
5
7
11
13
17
19
23
29
31
37
41
43
47
53
59
61
67
71
73
79
83
89
97
101
103
107
109
113
127
131
137
139
149
151
157
163
167
173
179
181
191
193
197
199
211
223
227
229
233
239
241
251
257
263
269
271
277
281
283
293
307
311
313
317
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541
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569
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587
593
599
601
607
613
617
619
631
641
643
647
653
659
661
673
677
683
691
701
709
719
727
733
739
743
751
757
761
769
773
787
797
809
811
821
823
827
829
839
853
857
859
863
877
881
883
887
907
911
919
929
937
941
947
953
967
971
977
983
991
997

5. **Prop.** *To find the prime factors of a given composite number.*[2]

As every composite number consists of the product of prime numbers, hence, if the given composite number be successively divided, as often as possible, by each of the prime numbers, beginning with the least by which the given number is divisible, until the last quotient is unity, the divisors will be the prime factors, which, when multiplied together, will reproduce the given composite number.

---

[1] If the natural numbers be formed into classes by tens, hundreds, &c., it will be found that the prime numbers do not occur in any regular order, either in the classes or in the order of the natural numbers.

[2] Resolve the number 32760 into its prime factors:

Here 32760 divided by 2 gives quotient 16380, or 2)32760
16880 ,, 2 ,, 8190 ,, 2)16380
8190 ,, 2 ,, 4095 ,, 2)8190
4095 ,, 3 ,, 1365 ,, 3)4095
1365 ,, 3 ,, 455 ,, 3)1365
455 ,, 5 ,, 91 ,, 5)455
91 ,, 7 ,, 13 ,, 7)91
13 ,, 13 ,, 1 ,, 13)13
1

The divisors 2, 2, 2, 3, 3, 5, 7, 13, are the prime factors of 32760.
And $32760 = 2\times2\times2\times3\times3\times5\times7\times13$, or $= 2^3\times3^2\times5\times7\times13$.

It is obvious that any composite number can only be resolved into one set of prime factors.

If two numbers be resolved into their prime factors, it will at once appear on inspection what factors are common to the two numbers; the product of these factors will be the greatest common measure of the two numbers.

Thus may be found the greatest common measure of 24 and 36:—

Here, $24 = 2 \times 2 \times 2 \times 3$, and $36 = 2 \times 2 \times 3 \times 3$,

The factors, 2, 2, 3, are common;

And, therefore, $2 \times 2 \times 3 = 12$, is the greatest common measure of 24 and 36.

The same method can be applied to find the least common multiple of two numbers.

In a similar way, if two numbers be resolved into their prime factors, by inspection may be seen what factors are common, and what factors are not common to the two numbers. The product of these factors will obviously be the least multiple of each of the two numbers, and consequently their least common multiple.

Thus may be found the least common multiple of 24 and 36:—

Here, $24 = 2 \times 2 \times 2 \times 3$ and $36 = 2 \times 2 \times 3 \times 3$.

The prime factors 2, 2, 3, are common, and 2, 3 are not common.

Hence the product $2 \times 2 \times 3 \times 2 \times 3$, or 72, of the prime factors $2 \times 2 \times 3$, which are common, and of 2 and of 3, which are not common, is the least common multiple of 24 and 36.

The resolution of numbers into their prime factors appears to be the more simple mode of discovering the greatest common measure and the least common multiple of two or more numbers. When the numbers are small this method can be applied with advantage. But when the given numbers are large there will arise some difficulty in discovering by inspection when a number is exactly divisible and when not divisible by a large prime number. This difficulty is obviated by a process by which can be determined with exactness the greatest common measure and the least common multiple of any numbers how great soever they may be.

6. Prop. *To find the greatest common measure, or the greatest common divisor of any two numbers.* (Euc. vii. 2.)

Let it be required to find the greatest common divisor of 189 and 224.

Here, 224 divided by 189 gives quot. 1, and rem. 35
189 ,, 35 ,, 5 ,, 14
35 ,, 14 ,, 2 ,, 7
14 ,, 7 ,, 2 ,, 0

189)224(1
189
35)189(5
175
14)35(2
28
7)14(2
14

The last divisor, 7, is a common measure of 189 and 224.

$$224 = 189 \times 1 + 35,\ \ 189 = 35 \times 5 + 14,$$
$$35 = 14 \times 2 + 7,\ \ 14 = 7 \times 2 + 0.$$

Here 7, the last divisor, measures 14, by the units in 2.

Next, 7 measures 14×2, a multiple of 14, and $14 \times 2 + 7$, or 35.

Thirdly, 7 also measures 35×5, and $35 \times 5 + 14$, or 189.

Fourthly, since 7 measures 189 and 85, its measure $189 \times 1 + 35$, or 224.

Therefore 7 is a common measure of 189 and 224.

And there is no number greater than 7 which will measure 189 and 227.

$$224 - 189 \times 1 = 35, \qquad 189 - 35 \times 5 = 14, \qquad 35 - 14 \times 2 = 7$$

For, if possible, let some number greater than 7 measure 189 and 224.

Then this number measures $224 - 189 = 35$, their difference.

It also measures 35×5, and $189 - 35 \times 5$ or 14.

And, thirdly, it measures $35 - 14 \times 2$ or 7.

That is, a number greater than 7 measures 7; which is impossible.

Wherefore 7, the last divisor, is the greatest common measure of 189 and 224.

Hence, if the greater of two numbers be divided by the less, and the preceding divisor always by the last remainder, until there be no remainder, the last divisor is the greatest common measure of the two numbers.

If the last divisor be unity, the two numbers are prime to each other, and have no common measure.

To find the greatest common measure of more numbers than two:—

1. Of three numbers.

Having found the G.C.M. of two of the three numbers, the G.C.M. of this number and the third number will be the G.C.M. of the three numbers.

2. Of four numbers.

Having found the G.C.M. of three of the numbers, the G.C.M. of this number and the fourth number will be the G.C.M. of the four numbers.

And in a similar way for five, six, seven, &c., numbers.

7. Prop.—*To find the least common multiple of two numbers.*

Let 18 and 24 be two numbers, of which the greatest common measure is 6.

$$\text{Then } 18 = 6 \times 3, \text{ and } 24 = 6 \times 4, \text{ also } 18 \times 24 = 6 \times 3 \times 6 \times 4,$$

And obviously the least common multiple of the two numbers will consist of the product of all the prime factors in the two numbers;

or the least common multiple of 18 and $24 = 6 \times 4 \times 3$ or $\frac{18 \times 24}{6} = 72$.

And there is no integral number less than 72, which is a less multiple of 18 and 24;

For 72 contains 18, 4 times, and 24, 3 times, and 3 and 4 being prime to each other;

Wherefore the L.C.M. of 18 and 24 $= \frac{18 \times 24}{6}$,

Or the least common multiple of two numbers, is equal to their product divided by their greatest common measure.

The following form is, perhaps, more convenient in practice.

L.C.M. of 18 and 24 $= \frac{18 \times 42}{6} = 18 \times \frac{24}{6} = 24 \times \frac{18}{6}$;

The least common multiple of two numbers is equal to the product of either of the numbers multiplied by the quotient arising from dividing the other by their greatest common measure.

If the two given numbers are prime to each other, their least common multiple is equal to the product of the numbers.

To find the least common multiple of more numbers than two:—

1. Of three numbers.

The L.C.M. of two of the numbers having been found, the L.C.M. of this number and the third number will be the L.C.M. of the three numbers.

2. Of four numbers.

The L.C.M. of three of the numbers having been found, the L.C.M. of this number and the fourth number will be the L.C.M. of the four numbers.

And so on for five, six, seven, &c., numbers.

When the several numbers are not large numbers, the process may be shortened by successive divisions of the given numbers, by prime factors which are common to two or more of the given numbers. By this means, all the divisors will consist of the common prime factors, and the numbers left after the divisions will be the factors which are not common to any two of the numbers. Then the product of the common prime factors, and the factors which are not common, will be the least common multiple of all the given numbers.[1]

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[1] Exemplify by finding the least common multiple of 15, 24, 36, and 42 by both methods.

First Method.

The G.C.M. of 24 and 36 is 12 : the L.C.M. $= \frac{24 \times 36}{12} = 72$.

„ G.C.M. of 72 and 42 is 6 : the L.C.M. $= \frac{72 \times 42}{6} = 504$.

„ G.C.M. of 504 and 15 is 3 : the L.C.M. $= \frac{504 \times 15}{3} = 2520$.

Second Method.

$$\begin{array}{r|l} 2 & 15,\ 24,\ 36,\ 42 \\ \hline 2 & 15,\ 12,\ 18,\ 21 \\ \hline 3 & 15,\ \ 6,\ \ 9,\ 21 \\ \hline & \ 5,\ \ 2,\ \ 3,\ \ 7 \end{array}$$

Here 2, 2, 3 are the common prime factors, and 5, 2, 3, 7 are the factors not common.

L.C.M. of 15, 24, 36, 42 $= 2 \times 2 \times 3 \times 5 \times 2 \times 3 \times 7 = 2520$.

EXERCISES.

I.

1. What is the distinction between odd and even numbers, and between prime and composite numbers?

2. Shew by examples that the sum and difference of any two odd numbers, however large, are always even numbers. Are the product and the quotient of two odd numbers always odd numbers?

3. When is one number said to be a measure of another? Give examples.

4. Explain what is meant by one number being a multiple or submultiple of another.

5. Explain what is meant by one number being prime to another.

6. When two numbers are prime to each other, are they necessarily prime numbers? Give examples.

7. What is meant by saying that one number is *a common measure* of two or more numbers? also, *the greatest common measure?*

8. When is a number *a common multiple* of two or more numbers, and when *the least common multiple?*

9. State the rules for finding the greatest common measure and least common multiple of two numbers.

10. If two numbers be respectively divided by their greatest common measure, what relation subsists between the two quotients? Is the relation true *in all cases?*

II.

Find the greatest common divisor of each of the following sets of two numbers:—

25 and 35: 72 and 96: 69 and 96: 531 and 135: 677 and 1788: 896 and 1344: 3458 and 3570: 6279 and 1495: 6111 and 1769: 20275 and 11390: 14133 and 16149: 13775 and 30218: 68635 and 19721: 58859 and 135894: 40033 and 129645: 29766 and 208362: 18769 and 253413: 236511 and 37499: 16897 and 58264: 4010401 and 4011203: 845315 and 265200.

III.

Find the greatest common measure of each of the following groups of numbers:—

805, 1311, and 1978: 720, 336, and 1736: 837, 1134, and 1377: 714, 289, and 629: 519, 1155, and 1617: 2142, 2184, and 4620: 1813, 2793, and 20286: 1680, 1440, 1960, and 1200: 376, 940, 1034, and 1081: 967765, 615155, 439895, and 989345.

IV.

Find the least common multiple of each of the following sets of numbers:—

12 and 18: 18 and 20: 20 and 35: 40 and 50: 90 and 95: 17 and 19: 319 and 407: 333 and 504: 222 and 189: 144 and 180: 250 and 520: 15875 and 43197: 30590 and 271469: 37499 and 236511: 173376 and 171072.

3, 4, and 5: 5, 9, and 11: 9, 12, and 16: 12, 18, and 20: 13, 17, and 96: 96, 105, and 180: 75, 185, and 216: 100, 101, and 103: 100, 240, and 515: 376, 1034, and 1081: 819, 1155, and 1617: 2142, 2184, and 4620: 3500, 8000, and 1780: 9, 10, 11, and 12: 24, 60, 81, and 99: 102, 195, 210, and 660: 2717, 2431, 3553, and 4199: 16, 27, 36, 42, and 45: 72, 120, 180, 24, and 36: 1001, 1309, 1547, 2431, and 1457: 15, 16, 18, 20, 24, 25, and 30: 2, 4, 6, 8, 10, 12, 14, 16, and 18: 24, 16, 6, 20, 8, 10, 30, 4, 25, and 12.

V.

1. Write all the common measures of 252 and 224, and all their common multiples which are less than 10000.

2. What is the least number that can be exactly divided by each of the nine digits?

3. Find the least number divisible by the following prime numbers: 3, 5, 7, 11, 13, 17, 19.

4. Find the least number divisible by each of the composite numbers, 6, 15, 20, 27, 64.

5. Determine the factors of the least common multiple of 573440, 244324080, and 3930927.

6. Find all the common multiples of 24, 15, 36, which lie between 1000 and 2000.

7. Find the least number which can be divided by 9, 12, 15, 18, with a remainder 3 in every case.

8. Find two numbers whose greatest common measure is 179, and least common multiple 56385.

9. Shew by examples whether the greatest common measure of two numbers can ever exceed the difference of the numbers; and their least common multiple can ever exceed their product.

10. Determine two numbers greater than 1000 that have 143 for their greatest common measure.

11. The greatest common measure of any two numbers is the least common multiple of all their common measures. Ex. 135135 and 145530.

12. Find the greatest number which will divide 5427 and 7245, and leave the remainders 7 and 5 respectively.

VI.

1. All the prime numbers except 2 and 5 end with the figures 1, 3, 7, or 9. Does this fact assist in distinguishing prime from composite numbers?

2. Express the composite numbers between 100 and 200 by the products of their prime factors.

3. Write down the prime numbers contained between 1 and 100 and between 900 and 1000.

4. Shew whether prime numbers occur in any regular order in each successive tens of the series of the natural numbers.

5. Decompose 831600 into its prime factors.

6. Find the factors of 6552 and of 4080, and their highest common factor.

7. Find the greatest common measure of 1071, 1092, 2310, by resolving the numbers into their prime factors.

8. Resolve 132288, 107328, 138216, and 97344, into their prime factors. And find their greatest common measure and their least common multiple.

9. The product of four consecutive numbers is 1680, find them.

10. When a series of numbers have been resolved into their prime factors, which of these factors must be taken to form by their product (1) the greatest common measure, (2) the least common multiple of the numbers. Form the greatest common measure and the least common multiple of 405, 570, 910.

11. Prove by any number of seven figures that when a number is divided by 9, it will have the same remainder as the sum of the digits divided by 9.

12. The product of the first and second of three numbers is 377, of the second and third 481, and of the first and third 1073; find the three numbers.

VII.

1. Every prime number greater than 5, increased or diminished by unity, is divisible by 6. Exemplify the truth of this by any ten prime numbers, and show whether the converse is true.

2. Find the prime factors of 1728, and shew in how many ways 1728 can be divided into two factors.

3. Determine whether either of the numbers 785432 and 785431 is divisible by 9.

4. Prove that one of the two numbers 23456789 and 23457698 is divisible by 11, and the other is not.

5. Shew how to find the number of divisors of a given composite number. What number of divisors has the least common multiple of 1428, 1287, 1560?

6. If the product of the natural numbers which precede any prime number be increased by unity, the sum is exactly divisible by that prime number. Ex. Shew that $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 + 1$ is exactly divisible by the prime number 11.

7. Any odd number not ending with 5 being given as a multiplicand; shew that it is always possible to find another number which

as a multiplier with the former will give a product composed entirely of the repetition of any one of the nine digits. Ex. Find numbers which, when multiplied by 13, shall produce a series of products, which respectively shall consist of repetitions of each of the nine digits.

8. If the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, be prefixed respectively to the figure 9 successively repeated, and these numbers so formed be each divided by 13; then (1) the quotients will consist of ten periods, each of six repeating figures: (2) the last three figures of each period will be the difference between 999 and the first three: and (3) the sums of the first and tenth periods, the second and ninth, the third and eighth, the fourth and seventh, and the fifth and sixth, will be respectively equal to one another. Shew the truth of these statements by performing the operations.

9. There are four prime numbers; the product of the first three is 2431; of the last three, 4199; of the first, second, and fourth, 2717; and of the first, third, and fourth, 3553; determine the four numbers.

10. There are three numbers of which the least common multiple of the first and second is 1125, of the first and third 1625, and of the second and third 2025; determine the three numbers.

VIII.

1. Find the least number of ounces of standard gold which can be coined into an exact number of half-sovereigns, if gold be worth £3 17s. $10\frac{1}{2}$d. per ounce.

2. Find the least number of pounds which can be paid in either half-crowns or guineas.

3. Find the least sum which can be paid in an integral number of francs and roubles, if a rouble be worth 3s. $4\frac{1}{2}$d., and a franc $9\frac{3}{4}$d.

4. If an American dollar is 4s. $3\frac{3}{4}$d., or $5\frac{1}{2}$ francs; what is the smallest sum which can be paid in either shillings, dollars, or francs?

IX.

1. What number of grains will be the *least* and the *greatest* common measure of 4oz. 10dwts. 12grs., and 10oz. 11dwts. 4grs. Troy?

2. Find the greatest common measure of 3cwts. 2qrs. 4lbs., and 3cwts. 3qrs. 12lbs.

3. What is the greatest number of ounces Avoirdupois that will exactly measure 15cwts. 1qr. 27lbs. 10oz., and 21cwts. 2qrs. 21lbs. 14oz.

4. If the pound Avoirdupois contain 7,000 grains, find *the greatest weight* which will measure both a pound Troy and a pound Avoirdupois; and *the least weight* which can be expressed without fractions in both pounds Troy and pounds Avoirdupois.

5. A cask is required which can be filled exactly by any one of the following measures, taken any number of times exactly: half a pint, half a gallon, three gallons, five gallons, and nine gallons; find the smallest cask for this purpose.

X.

1. The fore and hind wheels of a carriage are 12 and 15 feet in circumference; find the least number of revolutions of each which will give the same length.

2. In counting a number of pebbles, it was found that in telling them out by twos, threes, fours, fives, and nines, there were none left; what was the least possible number of the pebbles?

3. If the step of a man be 36 inches, a woman 24, and a boy 18, how many times will all three step together in walking 5 miles, supposing all three start together?

4. The sides of a triangular piece of ground are of lengths 45156, 52176, and 44532 feet respectively; find the length in feet of the longest chain which will exactly measure each of the three sides.

5. A shepherd on telling his sheep found that when he told them out by twos, threes, fours, and fives, he had none left, and he knew his flock was above 300 but less than 400. What was the number?

6. Find the greatest measure of length in terms of which both 1fur. 38yds. 1ft. 6in., and 1fur. 104yds. 1ft. 6in., can be expressed as integers.

XI.

1. Determine the least common multiple of 157 days 7hrs. 4min. 7sec., and 243 days 2hrs. 11min. 49sec.

2. For what unit of time will 23148 and 720720 seconds be represented by numbers prime to each other?

3. Find the greatest unit of time in which 15hrs. 12min., and 1 day 3hrs. 33min., can both be represented by integers.

4. Four bells toll at intervals of 3, 7, 12, 14 seconds respectively, and begin to toll at the same instant; when will they next toll together?

5. Four points, moving each at a uniform speed, take 198, 495, 891, 1155 seconds respectively, to describe the length of a given straight line. Supposing them to be together at any instant at the same end of the line, and to move in it from end to end continually, what interval of time will elapse before they are together at the same point again?

6. If three bodies move uniformly in similar orbits round the same centre in 87, 224, 365 days respectively: supposing all three in conjunction at a given time, find after how many days they will be first in conjunction again.

HINTS, RESULTS, ETC., FOR THE EXERCISES.

I. The first 8 questions may be answered from Art. 1. 9. Arts. 6, 7.

II. The following are the greatest common divisors of the pairs of numbers:

5 : 72 : 3 : 9 : 1 : 448 : 2 : 1 : 1 : 15 : 21 : 29 : 37 : 71 : 37 : 29766 : 7 : 11 : 1 : 73 : 5.

III. The following are the greatest common measures of each of the groups:

28 : 8 : 27 : 17 : 3 : 42 : 49 : 40 : 47 : 5.

IV. The following are the least common multiples of the successive sets of numbers:

36 : 180 : 140 : 200 : 1710 : 323 : 11803 : 18648 : 13986 : 720 : 13000 : 68035275 : 361053770 : 806265999 : 51493672.

60 : 495 : 144 : 180 : 24216 : 10080 : 199800 : 1040300 : 123600 : 190256 : 315315 : 6126120 : 498400 : 1980 : 35640 : 1021020 : $11\times13\times17\times19$: 15120 : $2^3\times3^2\times5$: $7\times11\times13\times17\times31\times47$: 7200 : 5040 : 1200.

V. 1. 2, 4, 7, 14, 28, are the common measures; and the least common multiple is 1216, and the largest under 10,000 is 9728. 2. 2250. 3. 14549535. 4. 8640. 5. $2^{14}.3^3.5.7^2.11^2.13.17.113$. 6. 1080, 1440, 1800. 7. Add 3 to the L.C.M. of the given numbers. 8. The quotient arising from dividing the L.C.M. by the G.C.M. of two numbers, consists of the product of two numbers prime to each other. The quotient in this case is 315, which is to be divided into two such numbers: find the numbers and verify the process. 9. Arts. 6, 7. 10. Any multiples greater than 6 of 143 will fulfil the conditions. 11. The G.C.M. is $3^3\times5\times7\times11$, from which all the common measures may be found, which are 31. 12. The greatest number is 20.

VI. 1. Art. 3. 2. Omit the 21 prime numbers between 100 and 200, and 79 composite numbers remain, which are readily resolved into their prime factors. 3. Employ the method of Art. 3. 4. Examine the series of prime numbers. 5. $2^4.3^3.5^2.7.11$. 6. The numbers when resolved into their prime factors are $2^3.3^2.7.13$, and $2^4.3.5.17$, in which the common factors are $2^3.3$. 7. 21. 8. The numbers when resolved are $2^6.3.13.53$, $2^6.3^2.13.43$, $2^3.3.13.443$, and $2^6.3^2.13^2$: of which the G.C.M. is $2^3.3.13$, and the L.C.M. is $2^6.3^2.13^2.43.53.443$. Art. 5. 9. Resolve 1680 into its prime factors. 10. See Art. 5. L.C.M. is $2.3^4.5.7.13.17$. 11. See Art. 2. 12. By means of the G.C.M. the numbers are found to be 29, 13, 37.

VII. 1. Take the prime numbers 13, 17, 31, 37, 41, 43, 61, 71, 211, 311. 2. Into twenty several different ways. 3. See Art. 2. 4. See Art. 2. 5. See Art. 5 and the note. 6. The truth of the statement may be verified by taking any other prime numbers. 7. The multipliers are respectively 8547, 17094, 25641, 34188, 42735, 51282, 59829, 68376, 76923. 8. The following are the first periods respectively of the ten quotients: 153846, 230769, 307692, 384615, 461538, 538461, 615384, 692307, 769230, 846153. The other two properties may be readily verified. 9. The numbers are 11, 13, 17, 19, which can be found by means of the G.C.M. 10. 125, 225, 325.

VIII. 1. The least number of ounces of standard gold is 80. 2. £21 can be paid by 20 guineas, or by 168 half-crowns. 3. The least sum is £2 3s. 10½d., which can be paid by 13 roubles or 54 francs. 4. The dollar, the franc, and the shilling respectively consist of 2277, 414, and 528 elevenths of a shilling.

IX. 1. 724 grains Troy. 2. 36lbs. Avoird. 3. 5554 ounces Avoird. 4. The greatest weight is 40 grains. The least 175lbs. Troy, or 144 Avoird. 5. 45 gallons.

X. 1. 4 revolutions of the larger wheel are equal to 5 of the smaller which can be made in running 60 feet. 2. 180 pebbles. 3. They stepped together 4440 times. The man took 8800 steps, the woman 13320, and the boy 17600. 4. 12 feet long. 5. 360 sheep. 6. 47 and 59 units; the unit being 16½ feet, or 198 inches.

XI. 1. 2674 days 9min. 59sec. 2. 23148 and 720720 seconds can be represented by 643 and 20020 when the unit of time is 36 seconds. 3. The greatest unit of time is 57 minutes. 4. In 2min. 48sec. 5. The interval will be 62370 seconds. The four points will have moved over the distance 315, 125, 70, 54 times respectively. 6. The three bodies will be in conjunction again after 7113120 days, when the first will have made 81760 revolutions in its orbit; the second, 31755; and the third, 19488.

ELEMENTARY ARITHMETIC,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION IX.

FRACTIONS.

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FRACTIONS.

Art. 1.—Def. 1. A fraction is an expression for a part or any number of parts of anything considered as an integer or whole.

A fraction is expressed by two numbers, one placed above the other with a line between them: the number placed below the line is called the denominator, and denotes the number of equal parts into which the integer is supposed to be divided; and the number above the line is called the numerator, and shews how many of these equal parts are expressed by the fraction.

Thus $\frac{3}{4}$ is a fraction, of which the denominator 4 denotes that the integer is to be divided into four equal parts, and the numerator 3 shews that three of these equal parts are expressed by the fraction.[1]

It follows from the definition of a fraction, that when the numerator is equal to the denominator, the value of the fraction is equal to unity; but if the numerator be less than the denominator, the value of the fraction is less than unity; and if the numerator be greater than the denominator, the value of the fraction is greater than unity.[2]

Def. 2. A fraction is called a *proper fraction* when the numerator is less than the denominator, and an *improper fraction* when the numerator is equal to or greater than the denominator.[3]

---

[1] Fractions may be supposed to originate by considering what part one number is of another. If parts of a number be taken, each part may be compared with the number itself. Thus, if the numbers 1, 2, 3 be compared with the number 4,

1 is one part of 4, or $\frac{1}{4}$ of 4,
2 is two parts or $\frac{2}{4}$ of 4,
3 is three parts or $\frac{3}{4}$ of 4.

The meaning of any fraction $\frac{3}{4}$ may be exemplified by assuming any concrete number whatever. Thus, if the integer or unit be £1 or 20 shillings, and this unit be divided into 4 equal parts, each part will consist of 5 shillings, and 3 of these parts will be 15 shillings, so that $\frac{3}{4}$ of £1 will be 15 shillings.

And again, if the integer be one lineal yard or 36 inches, when this integer is divided into 4 equal parts, each part will consist of 9 inches, and 3 of these parts will be 27 inches, wherefore $\frac{3}{4}$ of one yard will be 27 inches.

[2] In the series of fractions, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$, &c., each is less than the one which precedes it, and from the definition of a fraction it appears that the value of a fraction is *diminished* by *increasing* the denominator, while the numerator remains unaltered.

And in the series, $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, \frac{5}{4}, \frac{6}{4}$, &c., each is greater than the one which precedes it, or that the value of a fraction is *increased* by *increasing* the numerator, while the denominator remains constant.

[3] The word fraction is used both for the expression of any part or parts less than the whole, and for the whole itself, as well as for a number of parts greater than the whole. There is nothing in the nature of the case, nor in the manner in which fractions are written, to prevent the number of parts expressed being greater than the number of parts into which the unit is divided. But the word fraction will clearly

Def. 3. A mixed number consists of an integer and a fraction.

Def. 4. A simple fraction consists of one numerator and one denominator.

Def. 5. A compound fraction is defined to be the fraction of a fraction, or of more than two fractions.

Def. 6. A complex fraction is defined to be one that has an integer, a fraction, or a mixed number for its numerator or denominator.

Def. 7. An integer may be expressed in the form of a fraction by placing the integer for the numerator and unity for the denominator.

2. Prop. *A simple fraction may be considered as representing the quotient which arises from dividing the number above the line by the number below it.*[1]

For the expression $\frac{3}{4}$ has been assumed to indicate the quotient arising from the number 3 divided by 4.

It has also been assumed to denote three-fourths of any unit.

Now 3 divided by 4, or $\frac{1}{4}$ of 3, is three times as great as $\frac{1}{4}$ of 1.
It follows that 3 divided by 4 must be equal to $\frac{3}{4}$ of 1,
and the expression $\frac{3}{4}$ properly represents either of them.

3. Prop. *A fraction is multiplied by any number by multiplying the numerator by the number and retaining the denominator.*[2]

Thus any fraction $\frac{2}{15}$ multiplied by 7 gives $\frac{14}{15}$;
for in each of the fractions $\frac{2}{15}$ and $\frac{14}{15}$,
the unit is divided into fifteen equal parts;
and 14 of these parts are taken in one case and 2 in the other;
and since 14 parts are 7 times 2 parts,
it follows that the fraction $\frac{14}{15}$ is 7 times $\frac{2}{15}$,
or that $\frac{2}{15}$ multiplied by 7 gives $\frac{14}{15}$.

4. Prop. *A fraction is multiplied by any number by dividing the denominator by the number and retaining the numerator.*

Thus the fraction $\frac{3}{20}$ multiplied by 4 gives $\frac{3}{5}$;
for the unit in $\frac{3}{20}$ is divided into 20 equal parts,
and the unit in $\frac{3}{5}$ into 5 equal parts.

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in this case be applied in a sense different from that which expresses a number of parts less than the whole; as $\frac{5}{4}$ being greater than 1 or $\frac{4}{4}$, it would be absurd to speak of the former being a fraction of the latter. The meaning, however, of such an expression as $\frac{5}{4}$ is, that the unit is divided into 4 equal parts, and this expression $\frac{5}{4}$ denotes 5 of such equal parts; and this notation of fractions leads to no absurdity whatever. It cannot, however, be denied, that there is an inaccuracy of language, if a large quantity be stated to be a fraction of a smaller one.

[1] This may be illustrated by any concrete number whatever. Let the unit assumed be a lineal yard or 36 inches, then 3 units will contain 108 inches or equal parts, and the 3 units or 108 inches divided by 4 gives the quotient 27 inches. Again, $\frac{1}{4}$ of the unit or 36 inches will be 9 inches, and $\frac{3}{4}$ of the unit will be 27 inches. Hence it appears that 3 yards when divided by 4, or $\frac{1}{4}$ of 3 yards, is the same as $\frac{3}{4}$ of one yard.

[2] If a fraction be multiplied by a number equal to the denominator the product is equal to the numerator.

It follows that each of the equal parts in $\frac{3}{5}$ is 4 times as great as each of the equal parts in $\frac{3}{20}$;
and since 3, the same number of parts, is taken in each fraction,
therefore $\frac{3}{5}$ is 4 times $\frac{3}{20}$,
or $\frac{3}{20}$ multiplied by 4 gives $\frac{3}{5}$.

5. Prop. *A fraction is divided by any number by multiplying the denominator by the number and retaining the numerator.*

Thus $\frac{3}{5}$ divided by 4 gives $\frac{3}{20}$;
for the unit in $\frac{3}{5}$ is divided into 5 equal parts,
and the unit in $\frac{3}{20}$ is divided into 20 equal parts.
It follows that each of the equal parts in $\frac{3}{20}$ is $\frac{1}{4}$ of each of the equal parts in $\frac{3}{5}$;
and since 3, the same number of parts, is taken in each case,
therefore $\frac{3}{20}$ is $\frac{1}{4}$ of $\frac{3}{5}$,
or that $\frac{3}{5}$ divided by 4 gives $\frac{3}{20}$.

6. Prop. *A fraction is divided by any number by dividing the numerator by the number and retaining the denominator.*

Thus $\frac{14}{15}$ divided by 7 gives $\frac{2}{15}$;
for the unit in each of the fractions $\frac{14}{15}$ and $\frac{2}{15}$ is divided into 15 equal parts,
and 14 of these parts are taken in one case and 2 in the other;
and since 2 parts are one seventh of 14 parts,
it follows that $\frac{2}{15}$ is one seventh of $\frac{14}{15}$,
or that $\frac{14}{15}$ divided by 7 gives $\frac{2}{15}$.

7. Prop. *The numerator and denominator of a fraction may be multiplied by the same number without altering the value of the fraction.*[1]

For if the numerator of any fraction $\frac{3}{5}$ be multiplied by any number 4,
the fraction is multiplied by that number and becomes $\frac{12}{5}$;
and if the denominator of $\frac{12}{5}$ be multiplied by the same number 4,
the fraction is divided by that number and becomes $\frac{12}{20}$.
Now if a quantity be both multiplied and divided by the same number,
its value is not altered.

It follows, therefore, that if both the numerator and denominator of a fraction be multiplied by the same number, the value of the fraction is not altered.

---

[1] As any complex fraction as $\frac{2\frac{1}{3}}{3\frac{1}{4}}$ means no more than $2\frac{1}{3}$ is divided by $3\frac{1}{4}$, the reduction of such fractions rather falls under division of fractions. But every complex fraction may be reduced to a single fraction by multiplying the numerator and denominator by the least common multiple of the denominators, which occur in the numerator and denominator of the complex fraction: thus, $\frac{2\frac{1}{3}}{3\frac{1}{4}}$ becomes $\frac{28}{39}$, by multiplying the numerator and denominator by 12.

8. Prop. *The numerator and denominator of a fraction may be divided by the same number without altering the value of the fraction.*[1]

For if the numerator of a fraction $\frac{12}{20}$ be divided by any number 4,
the fraction is divided by that number and becomes $\frac{3}{20}$;
and if the denominator of $\frac{3}{20}$ be divided by the same number 4,
the fraction is multiplied by that number and becomes $\frac{3}{5}$.
Now if a quantity be both multiplied and divided by the same number, its value is not altered.

It follows, therefore, that if both the numerator and denominator of any fraction be divided by the same number, its value is not altered.

Hence it appears that the value of a fraction depends not on *the absolute*, but on *the relative* values of the numerator and denominator; and that by means of these two principles, fractions may be altered in form while they retain the same value.

9. Prop. *To convert a mixed number, or an integer and a fraction, into an improper fraction; and conversely.*

To convert a mixed number, as $3\frac{5}{6}$, to an improper fraction.
The mixed number $3\frac{5}{6}$ consists of 3 units and $\frac{5}{6}$ of a unit.
Here the unit being supposed to be divided into six equal parts,
the 3 units will be equal to 18 sixths of the unit,
and the 3 units and 5 sixths will be equal to 23 sixths;

that is, $3\frac{5}{6}$ or $3+\frac{5}{6}=\frac{18}{6}+\frac{5}{6}=\frac{18+5}{6}=\frac{3\times6+5}{6}=\frac{23}{6}$.

Hence a mixed number may be converted into an improper fraction by multiplying the integer by the denominator of the fraction and adding the numerator, and retaining the same denominator.

Conversely, to reduce a mixed number into an improper fraction.

To reduce the improper fraction $\frac{23}{6}$ to a mixed number.

Here $\frac{23}{6}=\frac{18+5}{6}=\frac{18}{6}+\frac{5}{6}=3+\frac{5}{6}$ or $3\frac{5}{6}$.

Hence if the numerator of an improper fraction be divided by the denominator, the quotient is the integral part, and the remainder (if any) is the numerator of the fractional part, the denominator remaining unchanged.

10. Prop. *To reduce a compound fraction to a single one.*

For if $\frac{2}{3}$ of $\frac{4}{5}$ be any compound fraction,
it implies that the fraction $\frac{4}{5}$ is to be divided into three equal parts,
and two of them are to be taken;
or that $\frac{4}{5}$ is to be divided by 3, and the result multiplied by 2.

---

[1] This principle of dividing the numerator and denominator by the same number may be applied to find the limits within which any given fraction lies.

Thus to show that $\frac{17}{257}$ lies between $\frac{1}{15}$ and $\frac{1}{16}$; 257 divided by 17 gives $15\frac{2}{7}$; and $15\frac{2}{7}$ is greater than 15 but less than 16; therefore $\frac{17}{257}$ or $\frac{1}{15\frac{2}{7}}$ is greater than $\frac{1}{16}$ but less than $\frac{1}{15}$, and consequently $\frac{17}{257}$ lies between $\frac{1}{16}$ and $\frac{1}{15}$.

Now $\frac{4}{5}$ divided by 3 gives $\frac{4}{15}$,
and $\frac{4}{15}$ multiplied by 2 produces $\frac{8}{15}$.
Hence $\frac{2}{3}$ of $\frac{4}{5}$ is equal to $\frac{8}{15}$.[1]

And in a similar way it may be shown that $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{9}$, or any compound fraction whatever, may be reduced to a single fraction. Hence any compound fraction is reduced to a single one by multiplying the numerators of the several fractions for a new numerator, and the denominators for a new denominator.

11. Prop. *To reduce fractions to lower, and to their lowest terms.*[2]

A fraction whose numerator and denominator are not prime to

---

[1] The compound fraction $\frac{2}{3}$ of $\frac{4}{5}$ is of the same value as $\frac{4}{5}$ of $\frac{2}{3}$, whatever be the value of the unit.

It has been shown above that $\frac{2}{3}$ of $\frac{4}{5}$ is equal to $\frac{8}{15}$. And $\frac{4}{5}$ of $\frac{2}{3}$ denotes that $\frac{2}{3}$ is to be divided into 5 equal parts, and 4 of them are to be taken, or that $\frac{2}{3}$ is to be divided by 5 and the result multiplied by 4.

Now $\frac{2}{3}$ divided by 5 gives $\frac{2}{15}$, and $\frac{2}{15}$ multiplied by 4 produces $\frac{8}{15}$. Therefore $\frac{4}{5}$ of $\frac{2}{3}$ is equal to $\frac{8}{15}$. Whence $\frac{2}{3}$ of $\frac{4}{5}$ is equal to $\frac{4}{5}$ of $\frac{2}{3}$.

[2] For example, the fraction $\frac{12}{18}$ is reduced to *lower* terms, $\frac{6}{9}$, by dividing the numerator and denominator by 2, a divisor of 12 and 18 : and $\frac{12}{18}$ is reduced to its *lowest* terms, $\frac{2}{3}$, by dividing the numerator and denominator by 6, the greatest divisor of 12 and 18.

It may happen that a fraction, after having been reduced to its lowest terms, may still be expressed by numbers too large and inconvenient for practical use; a series of fractions nearly equivalent in value and in lower terms may be found by converting the given fraction into a continued fraction, and reducing one, two, or more terms of the continued fraction to a simple fraction.

Def.—A continued fraction is one which has for its denominator an integer and a fraction; and which latter fraction has also for its denominator an integer and a fraction; and so on till the series terminates.

The following are examples of continued fractions, consisting of two, three, and four terms :—

$$\cfrac{1}{3+\cfrac{1}{7}}\quad,\quad\cfrac{1}{2+\cfrac{3}{4+\cfrac{5}{6}}}\quad,\quad\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11}}}}\quad,$$

The inconvenient manner in which the continued fraction runs across the page, led the late Sir John W. Herschell to adopt a method of writing the continued fraction in one horizontal line. The sign + being placed below the line which divides the terms of the fraction, indicates that the following fractional parts of the expression are each added to the denominators of the preceding fractions in the series. Accordingly the continued fractions above may be thus written :—

$$\frac{1}{3+}\frac{1}{7},\ \frac{1}{2+}\frac{3}{4+}\frac{5}{6},\ \frac{1}{5+}\frac{1}{7+}\frac{1}{9+}\frac{1}{11}.$$

To convert the fraction $\frac{179}{443}$ into a continued fraction :—

Here $\frac{179}{443}=\cfrac{1}{\frac{443}{179}}=\cfrac{1}{2+\frac{85}{179}}$ $\qquad$ $\frac{9}{85}=\cfrac{1}{\frac{85}{9}}=\cfrac{1}{9+\frac{4}{9}}$.

next $\frac{85}{179}=\cfrac{1}{\frac{179}{85}}=\cfrac{1}{2+\frac{9}{85}}$ $\qquad$ $\frac{4}{9}=\cfrac{1}{\frac{9}{4}}=\cfrac{1}{2+\frac{1}{4}}$

Hence by successive substitutions,

each other, can be expressed *in lower terms* by dividing the numerator and denominator by any common divisor; and into their *lowest terms* by dividing them by the greatest common measure or divisor.

12. PROP. *To reduce two or more fractions having different denominators to other equivalent fractions, each having the same common denominator.*

It will be always found most convenient in practice to employ the least common denominator.

1. Let the denominators be prime to each other and the fractions in their simplest form.

If there be two fractions:

Multiply the numerator and denominator of the first fraction by the denominator of the second; and the numerator and denominator of the second fraction by the denominator of the first.

But, if there be more than two:

Multiply the numerator and denominator of each fraction by the product of the denominators of the rest.

2. If the denominators be not prime to each other:

$\frac{179}{443}=\frac{1}{2+}\frac{1}{2+}\frac{1}{9+}\frac{1}{2+}\frac{1}{4}$, the equivalent continued fraction.

It will be obvious that the process above is the same as that of finding the greatest common divisor of the terms of the fraction, and that the successive quotients, 2, 2, 9, 2, 4, form the successive denominators of the continued fraction.

Conversely, the continued fraction $\frac{1}{2+}\frac{1}{2+}\frac{1}{9+}\frac{1}{2+}\frac{1}{4}$ may be reduced to a single fraction by reversing the process by which the continued fraction was found, begining with the last fraction in the series.

The successive simple fractions found by taking one, two, &c., terms of a continued fraction, form a series of fractions successively approximating to the value of the continued fraction.

Find the series of fractions each of which converges to the value of the continued fraction $\frac{1}{2+}\frac{1}{2+}\frac{1}{9+}\frac{1}{2+}\frac{1}{4}$.

One term	$\frac{1}{2}$	differs from $\frac{179}{443}$	by $\frac{85}{886}$ in excess.
Two terms $\frac{1}{2+}\frac{1}{2}=$	$\frac{2}{5}$	,,	by $\frac{9}{2215}$ in defect.
Three terms $\frac{1}{2+}\frac{1}{2+}\frac{1}{9}=$	$\frac{19}{47}$	,,	by $\frac{4}{20821}$ in excess.
Four terms $\frac{1}{2+}\frac{1}{2+}\frac{1}{9+}\frac{1}{2}=$	$\frac{40}{99}$	,,	by $\frac{1}{43857}$ in defect.

And the whole five terms $\frac{1}{2+}\frac{1}{2+}\frac{1}{9+}\frac{1}{2+}\frac{1}{4}=\frac{179}{443}$, the total value of the continued fraction.

The third converging fraction $\frac{19}{47}$ may be found by a very simple process. Since $19=9\times2+1$, and $47=9\times5+2$; it appears that 19, the numerator of the third fraction, is equal to the product of 2, the numerator of the second fraction, and 9, the third quotient, added to 1, the numerator of the first fraction: and 47, the denominator of the third fraction, is equal to the product of 5, the denominator of the second fraction, and 9, the third quotient, added to 2, the denominator of the first fraction. And in a similar way may be found the fourth, &c., converging fractions.

Find the least common multiple of all the denominators.
Divide this number by each of the denominators respectively,
and then multiply the numerator and denominator of each of the fractions given by its corresponding quotient.[1]

An integer may be reduced to a fraction with any given denominator by writing the integer as a fraction with unity for its denominator, and then multiplying the numerator and denominator by the given denominator.

13. Prop. *To find the sum of two or more fractions.*[2]

If the fractions have the same common denominator, their sum will be expressed by the sum of the numerators taken for a new numerator and placed over the common denominator.

When the fractions have not the same denominator, when reduced to their least common denominator their sum can be found.

In case of mixed numbers, in practice, it will be found most convenient first to find the sum of the fractional, next of the integral parts, and then the sum of both will be the sum required.

Instead of adding all the fractions together by one operation, it will sometimes be found convenient first to find the sum of two of the fractions, next to add a third fraction to this sum, and so on until all the fractions have been added.

14. Prop. *To find the difference of two fractions.*

If the fractions have a common denominator, their difference will be expressed by the difference of the numerators placed over the common denominator.

If the fractions have different denominators, they must be reduced to their least common denominator, and then their difference can be found.

Instead of reducing mixed numbers to improper fractions, and then finding their difference, it will be found more convenient in practice to reduce the fractional parts only to a common denominator, and to find first the difference of the fractional parts, next of the integral parts.[3]

---

[1] Two or more fractions may be compared, or their relative values ascertained, by reducing them to the same common denominator.

It may be shown, in general, that if any number be added to, or subtracted from, the numerator and denominator of a fraction, the value of the fraction is either increased or decreased in value.

[2] In the operations of addition and subtraction of integers, the units must all be of the same magnitude, in order that the sum and difference of any given numbers may be expressed by one number: so also in fractions, the parts of the unit expressed by each fraction must be of the same magnitude before the sum or difference of any fractions can be expressed as one fraction. If each of the given fractions express different parts of the unit, they must be converted to others of equivalent values, expressing the same parts of the unit; or in other words, be reduced to the same common denominator.

[3] If the numerator of the fractional part to be subtracted be greater than that

15. Prop. *To find the product of two or more fractions.*[1]

Def. To multiply one fraction by another, is to take such part or parts of one as are expressed by the other; and this is effected by dividing the former fraction by the denominator, and multiplying it by the numerator of the latter fraction.

To find the product of $\frac{3}{5}$ by $\frac{4}{7}$.

Here the multiplication of $\frac{3}{5}$ by $\frac{4}{7}$ implies that $\frac{3}{5}$ is to be divided by 7, and the result multiplied by 4.

Now $\frac{3}{5}$ divided by 7 gives $\frac{3}{5\times7}$;

and this result, $\frac{3}{5\times7}$, multiplied by 4, produces $\frac{3\times4}{3\times7}$;

and, therefore, $\frac{3}{5}\times\frac{4}{7}=\frac{3\times4}{5\times7}$, or $\frac{12}{35}$.

Hence the product of two fractions is found by multiplying the two numerators for the numerator, and the denominators for the denominator of the product.

In the same way it may be shewn that the product of more fractions than two is found by multiplying all the numerators together, and all the denominators together, for the numerator and denominator of the product.

16. Prop. *To divide one fraction by another.*[2]

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from which it is to be taken, the subtraction is not possible. But the subtraction can be made possible by increasing the latter by a fraction equal to unity, and the other by unity. Thus in finding the difference of $15\frac{3}{7}$ and $8\frac{5}{7}$; $\frac{5}{7}$ cannot be taken from $\frac{3}{7}$, but if $\frac{7}{7}$ be added to $15\frac{3}{7}$, and 1 to $8\frac{5}{7}$, the subtraction then becomes possible, as "the difference of two numbers is not altered when each of the numbers is equally increased."

[1] In the multiplication and division of fractions, it will generally be found most convenient, first to reduce mixed numbers and compound fractions to simple fractions.

There is an impropriety of language in the employment of the word *multiply* in reference to proper fractions. To multiply two proper fractions is to take of neither of them so much as once, but only that part of one fraction which is expressed by the other: since a proper fraction is always less than unity, it follows that the product of two proper fractions will be less than either of them, and the multiplication produces *decrease* and not *increase* of magnitude.

In practice, however, it will be found more convenient in the multiplication of fractions to divide any factors of the numerator and denominator which are by the same numbers divisible, and to use the quotients instead of them, and the product will be expressed in its lowest terms.

Thus to multiply $\frac{1}{3}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{9}{20}$ together,

$$\text{here } \frac{1}{3}\times\frac{5}{6}\times\frac{6}{7}\times\frac{9}{20}=\frac{1\times1\times1\times3}{1\times1\times7\times4}=\frac{3}{28}.$$

First, 3 in denominator and 9 in numerator are divisible by 3; next, 5 in numerator and 20 in denominator are divisible by 5; thirdly, 6 in denominator and 6 in numerator are divisible by 6, and the respective quotients are used instead of the given numerators and denominators. If, however, the numerators are multiplied for the new numerator, and the denominators for the new denominator, the fraction is $\frac{270}{840}$, which, when reduced by 90 the greatest common measure, becomes $\frac{3}{28}$.

[2] The quotient of one fraction divided by another may be found by dividing the

Def. To divide one fraction by another is to find how many times or parts of a time the latter is contained in the former.

Since the operation of division is the reverse of that of multiplication, and it has been shown that the product of one fraction by another is effected by dividing the former fraction by the denominator and multiplying it by the numerator of the latter, it follows that one fraction will be divided by another by dividing the former fraction by the numerator and multiplying it by the denominator of the latter fraction.

Thus, to divide $\frac{4}{5}$ by $\frac{5}{6}$,
the fraction $\frac{4}{5}$ must be divided by 5, and multiplied by 6.
Now $\frac{4}{5}$ divided by 5 gives $\frac{4}{5\times5}$,
and $\frac{4}{5\times5}$ multiplied by 6 produces $\frac{4\times6}{5\times5}$;
therefore $\frac{4}{5}\div\frac{5}{6}=\frac{4\times6}{5\times5}$, or $\frac{4}{5}\times\frac{6}{5}$.

That is, the quotient of one fraction divided by another is found by multiplying the dividend by the reciprocal of the divisor.

Or the division of one fraction by another may be defined to be a method to find a third fraction, called the quotient, such that the product of the divisor and quotient shall be equal to the dividend.

To divide $\frac{4}{5}$ by $\frac{5}{6}$.

From the nature of division,
the product of the divisor and quotient is equal to the dividend,
and in this case $\frac{5}{6}\times$ quotient $=\frac{4}{5}$.
Multiply those equals by $\frac{6}{5}$;
then $\frac{6}{5}\times\frac{5}{6}\times$ quotient $=\frac{4}{5}\times\frac{6}{5}$;
or $\frac{30}{30}\times$ quotient $=\frac{4}{5}\times\frac{6}{5}$; but $\frac{30}{30}=1$.
Therefore the quotient, or $\frac{4}{5}\div\frac{5}{6}=\frac{4}{5}\times\frac{6}{5}$;
or the quotient is found by multiplying the dividend by the reciprocal of the divisor.

17. Prop. *To find the value of a fraction of any concrete quantity, and conversely.*[1]

If the units of the concrete quantity be divided by the denominator of the fraction, the quotient expresses one part of the fraction; and if this quotient be multiplied by the numerator, the product will express the value of the fractional parts required.

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numerator and denominator of the dividend by the numerator and denominator of the divisor respectively. But this is very seldom possible so as to obtain a simple fraction in the quotient. The difficulty may be obviated by first reducing the two fractions to a common denominator.

Thus, to divide $\frac{4}{5}$ by $\frac{5}{6}$,
Here $\frac{4}{5}=\frac{4}{5}\times\frac{6}{6}=\frac{24}{30}$, and $\frac{5}{6}=\frac{5}{6}\times\frac{5}{5}=\frac{25}{30}$.
Then $\frac{4}{5}\div\frac{5}{6}=\frac{24}{30}\div\frac{25}{30}=\frac{24}{25}$.

[1] Any number of concrete units may be considered as a part of any other number of concrete units of the same kind: as 5 shillings may be considered as a part of 12 shillings. Here the 12 shillings is the unit divided into 12 parts, and 5 parts

Conversely, to reduce any concrete quantity or fraction to the fraction of any other unit.[1]

First: Any fraction of a less unit is reduced to the fraction of a greater unit by *dividing* the fraction of the less unit by the number of less units which make one of the greater units.

Secondly: A fraction of a greater unit is reduced to one of a less unit by *multiplying* the fraction by the number of less units in the greater unit.

18. Prop. *To find the sum or difference of concrete fractions.*

The sum or difference of any concrete fractions can be found by reducing the given fractions to fractions of the same unit, and then finding their sum or difference.

Or the value of each of the concrete fractions may be first found in smaller units of the same kind, and then the sum or difference may be found of these units.

19. Prop. *To find the product or quotient of concrete fractions.*

The method of multiplication and division of concrete fractions differs not from that of abstract fractions.

If a concrete fraction be multiplied or divided by an abstract fraction, the product or quotient will be parts of the same concrete unit as the multiplicand or dividend.

If both fractions be concrete, the nature of the product or quotient will be determined by the consideration, whether the results of these operations do, or do not admit of any rational interpretation.

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are to be taken; it will then be clear that 5 shillings is $\frac{5}{12}$ of the unit, or of 12 shillings.

Find the value of $\frac{5}{8}$ of one pound sterling.

$$\begin{array}{rrrl} & \textit{s.} & \textit{d.} & \\ 8) & \underline{20} & \underline{0} & = £1 \\ & 2 & 6 & = \tfrac{1}{8} \text{ of } £1 \\ & & \underline{5} & \\ & \underline{12} & \underline{6} & = \tfrac{5}{8} \text{ of } £1. \end{array}$$

[1] Conversely, what fraction of £1 is 12s. 6d.?

Here 12s. 6d. = 25 sixpences,
and £1 = 20s. = 40 ,,

It is obvious that 12s. 6d. will be the same part of £1
as 25 sixpences is of 40 sixpences,
that is, 12s. 6d. = $\frac{25}{40}$ or $\frac{5}{8}$ of £1.

Reduce $\frac{5}{27}$ of one pound to the fraction of a shilling.

Here 20 of the smaller units make one of the larger.
Thus $\frac{1}{27}$ of £1 $= \frac{1}{27} \times 20 = \frac{20}{27}$ of 1 shilling.

Conversely, Reduce $\frac{20}{27}$ of 1 shilling to the fraction of a pound.

Here 1 larger unit is equal to 20 of the smaller,
And $\frac{20}{27}$ of 1 shilling $= \frac{20}{27} \div 20 = \frac{20}{27} \times \frac{1}{20} = \frac{1}{27}$ of £1.

EXERCISES.

Abstract Fractions.

I.

1. Define a fraction, and explain the notation of fractions.

2. What is meant by an improper fraction and a mixed number? How are improper fractions converted into mixed numbers, and conversely?

3. Convert the improper fractions $\frac{9}{2}$, $\frac{17}{3}$, $\frac{61}{8}$, $\frac{1711}{12}$, $\frac{323923}{56}$, to mixed numbers: and conversely; reduce the mixed numbers, $4\frac{1}{2}$, $5\frac{2}{3}$, $7\frac{5}{8}$, $142\frac{7}{12}$ and $5784\frac{19}{56}$ to improper fractions.

4. Simplify $\frac{12+9+6}{3}$ and $\frac{12\times9\times6}{3}$, and find what part the former is of the latter.

5. What are the advantages in arithmetical operations of employing fractions expressed by the smallest numbers possible? State how fractions expressed by large numbers may be reduced to equivalent fractions expressed by smaller numbers. Is this always possible?

6. State the rules for changing any given fractions to others of equal value, but having the least common denominator.

7. Explain why fractions having different denominators must be altered in form before their sum or difference can be expressed by one fraction.

8. Define the multiplication of two integers and the multiplication of two fractions, and show how the multiplication of two fractions is not inconsistent with the multiplication of two integers.

9. How is it shown that the product of any two fractions as $\frac{3}{5}$ and $\frac{7}{8}$ is equivalent to either of the compound fractions $\frac{3}{5}$ of $\frac{7}{8}$ or $\frac{7}{8}$ of $\frac{3}{5}$?

10. Explain how it is that the product of two proper fractions gives a fraction less than either of them: and the quotient, a fraction greater than either of them.

11. Show how the rule for the division of one fraction by another may be deduced from the method of the multiplication of one fraction into another.

12. Of the eleven numbers less than 12, which are aliquot parts of 12? Show how those numbers which are not aliquot parts of 12, may be expressed as the sum of two or more aliquot parts.

II.

Reduce the following fractions to equivalent fractions, each having the least possible numerator and denominator:—

$\frac{209}{380}$, $\frac{825}{960}$, $\frac{216}{3186}$, $\frac{1155}{168}$, $\frac{7854}{2160}$, $\frac{3300}{4235}$, $\frac{2993}{3321}$, $\frac{11385}{16335}$, $\frac{42237}{75582}$, $\frac{16637}{24257}$,
$\frac{14628}{21041}$, $\frac{217800}{245025}$, $\frac{243936}{313632}$, $\frac{135135}{145530}$, $\frac{140971}{487621}$, $\frac{87968}{277288}$, $\frac{294025}{3059675}$.

III.

Reduce the following sets of fractions to equivalent fractions, each set having the least common denominator.

1. $\frac{1}{3}$ and $\frac{1}{5}$; $\frac{2}{3}$ and $\frac{3}{4}$; $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{7}$; $\frac{2}{5}$, $\frac{4}{7}$ and $\frac{5}{12}$; $\frac{5}{7}$, $\frac{3}{8}$, $\frac{4}{9}$ and $\frac{11}{13}$; $\frac{3}{5}$, $\frac{5}{8}$, $\frac{4}{9}$ and $\frac{8}{11}$.

2. $\frac{2}{3}$ and $\frac{4}{9}$; $\frac{4}{9}$, $\frac{2}{3}$ and $\frac{7}{18}$; $\frac{5}{9}$, $\frac{1}{2}$, $\frac{1}{12}$ and $\frac{5}{36}$; $\frac{2}{5}$, $\frac{3}{4}$, $\frac{7}{20}$ and $\frac{51}{100}$; $\frac{4}{25}$, $\frac{3}{8}$, $\frac{7}{40}$, $\frac{2}{125}$ and $\frac{3}{1000}$.

3. $\frac{3}{20}$ and $\frac{7}{25}$; $\frac{1}{20}$, $\frac{1}{25}$ and $\frac{1}{24}$; $\frac{5}{8}$, $\frac{7}{12}$ and $\frac{11}{18}$; $\frac{5}{8}$, $\frac{7}{12}$, $\frac{1}{7}$ and $\frac{1}{3}$; $\frac{1}{2}$, $\frac{5}{12}$, $\frac{2}{15}$, $\frac{7}{8}$ and $\frac{5}{16}$.

IV.

Find the sums of the following twenty-three sets of fractions in their simplest forms:—

1. $\frac{3}{5}$ and $\frac{1}{5}$. 2. $\frac{3}{5}$ and $\frac{4}{7}$. 3. $2\frac{2}{3}$ and $5\frac{2}{7}$. 4. $\frac{11}{15}$ and $54\frac{7}{8}$. 5. $\frac{2}{3}$, $\frac{11}{17}$ and $\frac{4}{27}$. 6. $1\frac{1}{2}$, $2\frac{1}{3}$ and $3\frac{1}{4}$. 7. $\frac{3}{4}$ of $\frac{5}{6}$ and $\frac{3}{5}$ of $2\frac{1}{2}$. 8. $\frac{2}{7}$, $4\frac{1}{3}$ and $\frac{3}{7}$ of 2. 9. $\frac{10}{26}$, $\frac{12}{35}$, $\frac{17}{130}$, $\frac{22}{45}$ and $\frac{16}{63}$. 10. $100\frac{2}{5}$, $64\frac{5}{9}$ and $\frac{3}{7}$ of 701. 11. $\frac{5}{9}$ of $\frac{27}{40}$, $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{14}{15}$ and $\frac{3}{5}$ of $\frac{5}{7}$ of $\frac{7}{9}$. 12. $\frac{2}{99}$, $\frac{51}{15}$, $\frac{96}{504}$, $\frac{105}{231}$ and $\frac{91}{520}$. 13. $253\frac{1}{2}$, $47\frac{1}{7}$, $5\frac{1}{3}$, and $\frac{2}{3}$ of $7\frac{1}{2}$. 14. $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$ and $\frac{2}{3}$ of 3704. 15. $6\frac{1}{5}$, $8\frac{1}{7}$, $9\frac{1}{11}$ and $10\frac{1}{13}$. 16. $\frac{3}{8}$ of 5, $\frac{4}{7}$ of 6 and $\frac{5}{11}$ of 7. 17. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ and $\frac{8}{9}$. 18. $10\frac{7}{2431}$, $20\frac{29}{2717}$, $30\frac{589}{3553}$ and $40\frac{3446}{4199}$. 19. $\frac{3}{4}$ of $\frac{5}{7}$ of $52\frac{4}{15}$, $\frac{2}{5}$ of $\frac{2}{9}$ of $506\frac{1}{4}$ and $\frac{3}{5}$ of $\frac{5}{12}$ of $\frac{3}{8}$ of 1864. 20. $\frac{5}{12}$, $\frac{5}{12}$ of $\frac{4}{11}$, $\frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10}$, $\frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10}$ of $\frac{2}{9}$, and $\frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10}$ of $\frac{2}{9}$ of 8. 21. $\frac{3}{17}$ of $2\frac{5}{57}$ and $\frac{7}{19}$ of $\frac{2}{9}$ of $12\frac{3}{14}$. 22. $\frac{816}{166463}$ and $\frac{1}{1393}$. 23. $13\frac{13}{63}$, $14\frac{17}{42}$ and $15\frac{11}{21}$.

V.

Find the differences between each of the following two fractions:—

1. $\frac{7}{8}$ and $\frac{5}{12}$. 2. $3\frac{1}{6}$ and $5\frac{1}{5}$. 3. $3\frac{1}{5}$ and $5\frac{1}{6}$. 4. $2\frac{5}{7}$ and $12\frac{3}{4}$. 5. $\frac{4}{5}$ of $140\frac{1}{2}$ and $\frac{1}{8}$ of 1000. 6. $\frac{2}{3}$ of $\frac{3}{5}$ of $7\frac{1}{2}$ and $\frac{5}{8}$ of 200. 7. $\frac{1}{9}$ of $\frac{6}{15}$ of $\frac{20}{32}$ and $\frac{5}{6}$ of $\frac{4}{9}$ of $\frac{12}{30}$. 8. $101\frac{7}{12}$ and $\frac{3}{4}$ of $\frac{5}{11}$ of 999. 9. $\frac{5}{8}$ of $\frac{4}{10}$ and $\frac{7}{9}$ of $\frac{4}{21}$. 10. $\frac{3}{5}$ of $\frac{2}{11}$ of 25 and 10. 11. $\frac{2}{3}$ of $\frac{5}{8}$ of $16\frac{2}{7}$ and $49\frac{1}{50}$. 12. $\frac{7}{8}$ of $\frac{11}{12}$ of 2509 and $\frac{1}{3}$ of $\frac{1}{2}$ of 10000. 13. $4\frac{1}{3}$ of $5\frac{3}{8}$ and $2\frac{1}{7}$ of $\frac{14}{15}$. 14. $\frac{35}{91}$ of 9999 and $\frac{13}{36}$ of 10000. 15. $\frac{7}{8}$ of $\frac{1}{11}$ of 7874 and $\frac{7}{9}$ of $5896\frac{1}{7}$.

VI.

Find the products of the following sets of fractions in their simplest form:—

1. $\frac{1}{2}$ and $\frac{1}{3}$. 2. $\frac{1}{3}$ and $\frac{1}{4}$. 3. $\frac{2}{3}$ and $\frac{3}{4}$. 4. $\frac{5}{6}$ and $\frac{6}{7}$. 5. $\frac{3}{7}$ and $\frac{5}{11}$. 6. $1\frac{1}{2}$ and $2\frac{1}{3}$. 7. $20\frac{1}{4}$ and $12\frac{4}{9}$. 8. $\frac{1}{3}$, $\frac{1}{7}$ and $\frac{1}{8}$. 9. $2145\frac{19}{21}$ and $798\frac{7}{9}$. 10. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$. 11. $\frac{1}{6}$, $\frac{4}{51}$, $\frac{5}{24}$ and $\frac{3}{25}$. 12. $\frac{2}{3}$ of $\frac{5}{7}$ and $\frac{1}{2}$ of $\frac{1}{4}$. 13. $\frac{3}{5}$ of $2\frac{1}{2}$ and $\frac{5}{7}$ of $4\frac{1}{8}$. 14. $\frac{2}{7}$ of $\frac{7}{22}$ and 11. 15. $\frac{1}{3}$ of $\frac{1}{7}$ of 2 and $\frac{1}{8}$ of $\frac{2}{5}$ of 1000. 16. $\frac{2}{3}$ of $\frac{3}{7}$ of $23\frac{1}{10}$ and $\frac{4}{11}$ of $\frac{5}{9}$ of $290\frac{1}{20}$. 17. $\frac{5}{8}$ of $\frac{3}{5}$ of 1864 and $\frac{3}{4}$ of $\frac{3}{5}$ of 11. 18. $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, $4\frac{1}{5}$ and $5\frac{1}{6}$. 19. $\frac{5}{7}$ of $\frac{428571}{714285}$ and $\frac{1}{12}$ of $\frac{307692}{230769}$. 20. $\frac{15}{64}$, $\frac{16}{27}$, $\frac{11}{10}$, $\frac{72}{385}$, and $\frac{21}{85}$. 21. $7\frac{7}{8}$, $5\frac{1}{3}$, $2\frac{1}{6}$, and $\frac{14}{15}$. 22. $\frac{11}{12}$, $\frac{17}{18}$, $\frac{29}{30}$, $\frac{47}{48}$ and $\frac{59}{60}$. 23. $1000\frac{1}{100}$ and $100\frac{1}{1000}$. 24. $1\frac{5}{11}$ of 1000 and $6\frac{1}{4}$ of 10000.

VII.

Find the respective quotients arising from the following dividends and divisors:—

1. $\frac{2}{3}$ by $\frac{5}{7}$. 2. 8 by $2\frac{1}{3}$. 3. $3\frac{1}{2}$ by $4\frac{1}{5}$. 4. $2\frac{1}{2}$ by $\frac{1}{3}$ of $6\frac{3}{4}$. 5. $\frac{1}{3}$ of $\frac{3}{4}$ by $\frac{5}{7}$ of $3\frac{11}{15}$. 6. $\frac{1}{5}$ of $\frac{1}{55}$ by $\frac{1}{7}$ of $\frac{1}{77}$. 7. $\frac{1}{2}$ of $\frac{1}{4}$ of $\frac{1}{6}$ by $\frac{1}{3}$ of $\frac{1}{5}$ of $\frac{1}{7}$. 8. $\frac{2}{3}$ of 21 by $\frac{3}{5}$ of 15. 9. $\frac{5}{6}$ of $\frac{7}{8}$ of $3\frac{1}{5}$ by $\frac{2}{3}$ of $\frac{5}{6}$ of $4\frac{1}{5}$. 10. $\frac{7}{20}$ of $\frac{5}{24}$ by $\frac{4}{25}$ of $\frac{5}{24}$. 11. $1215\frac{7}{12}$ by $512\frac{3}{7}$. 12. $\frac{7}{11}$ of $\frac{5}{12}$ of 10000 by $\frac{1}{100}$ of $3\frac{1}{7}$.

VIII.

1. How may the relative magnitudes of two or more fractions be compared? Express by the least integers the relative magnitudes of the fractions $\frac{11}{19}$, $\frac{18}{29}$, and $\frac{23}{38}$.

2. Express 11, 13, 17, as fractions, each with the same denominator, 23.

3. What part of 15 is $\frac{5}{27}$ of $4\frac{1}{2}$?

4. Arrange the fractions $\frac{7}{15}$, $\frac{8}{35}$, $\frac{23}{44}$, $\frac{29}{56}$, in order of magnitude.

5. Change the fraction $\frac{19}{23}$ into one whose numerator is 209; and $\frac{25}{27}$ into one whose denominator shall be 729, so that the values of the two fractions may retain the same value.

6. Which is the greater of the two of the following sets of fractions—$\frac{20}{17}$ or $\frac{25}{23}$; $\frac{4}{5}$ of $\frac{5}{6}$ or $\frac{3}{4}$ of $\frac{4}{5}$; $\frac{2}{3}$ of $1\frac{1}{2}$ or $\frac{5}{6}$ of $1\frac{1}{5}$, and $\frac{2}{3}$ of $\frac{7}{10}$ or $\frac{6}{7}$ of $\frac{21}{34}$?

7. Find the alteration produced in the value of a proper fraction $\frac{7}{12}$, first by adding, next by subtracting, the same number 3 from the numerator and denominator.

8. If the number 5 be added to the numerator and denominator of any *improper* fraction $\frac{13}{11}$, or taken from it, when is the value of the fraction increased or diminished?

9. Show that the fraction $\frac{2+3}{5+7}$ lies between $\frac{2}{5}$ and $\frac{3}{7}$.

10. Show that the fractions $\frac{42+7}{54+9}$ and $\frac{42-7}{54-9}$, are equal to each of the fractions $\frac{7}{9}$ or $\frac{42}{54}$, and explain the reason.

11. The fraction $\frac{2+4+6}{3+5+7}$ lies between the greatest and least of the fractions $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$.

12. If any number of fractions be equal, then any one of them is equal to the fraction whose numerator is equal to the sum of all the numerators, and whose denominator is equal to the sum of all the denominators. Exemplify this in the case of six equal fractions.

IX.

1. Show that the sum of $\frac{1}{5}$ of $3\frac{3}{4}$, and $\frac{2}{9}$ of $1\frac{1}{8}$ is equal to 1.

2. To 479 add $1\frac{904}{905}$, and repeat the addition six times.

3. Add together the greatest and least of the fractions $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{19}{20}$, and subtract this sum from the sum of the other two fractions.

4. Change the fractions $\frac{37}{46}$, $\frac{78}{97}$, and $\frac{115}{143}$, into fractions having a

common denominator: and express the difference of the first two as a fraction of the difference of the second two.

5. Add together $\frac{12}{91}$, $\frac{6}{77}$, and $\frac{9}{143}$, and find what is the least fraction with denominator 1000, which must be added in order that the sum may be greater than unity.

6. Show that the sum of $\frac{2}{3}$ and $\frac{4}{7}$ is 13 times their difference.

7. What fraction subtracted from the sum of $\frac{3}{4}$ and $\frac{5}{7}$ will have unity for the remainder?

8. Subtract the sum of $\frac{5}{6}$ of $\frac{3}{10}$, $4\frac{3}{8}$, and $\frac{1}{4}$ of $1\frac{2}{3}$ from 12.

9. By how much does the sum of $1\frac{1}{26}$ and $\frac{9}{13}$ exceed the difference? What fraction of 7 is the excess?

10. What is that number of which the sum of one-third, one-fourth, and one-fifth, is equal to 235?

11. Subtract $\frac{17}{200}$ four times from $\frac{197}{200}$, and state how many times it may be subtracted, and what is the last remainder.

12. Find three fractions whose numerators shall be 3, 5, 7 respectively, and their sum equal to unity.

X.

1. Show that the quotient of $\frac{7}{8}$ divided by $\frac{1}{3}$, is 9 times their product.

2. What is the difference and the product of the quotients of $\frac{3}{5}$ divided by $\frac{4}{7}$, and $\frac{4}{7}$ divided by $\frac{3}{5}$?

3. What fraction divided by $\frac{2}{3}$ of $\frac{4}{5}$ of $1\frac{1}{2}$ will give the quotient unity?

4. What fraction multiplied by $\frac{5}{7}$ of $\frac{3}{11}$ will make the product equal to 2?

5. How often does $\frac{9}{17}$ of the sum of $\frac{2}{11}$ and $\frac{15}{8}$ diminished by $\frac{1}{36}$ contain $\frac{4}{97}$?

6. Multiply the sum of $\frac{3}{4}$ and $\frac{5}{12}$ by their difference, and find how much the product is less than unity.

7. Find the product and the quotient of the sum of $\frac{2}{3}$ and $\frac{1}{5}$ and the difference of unity and $\frac{1}{13}$: and ascertain which result is the greater.

8. What is the quotient which arises from dividing the difference between the first and second, by the sum of the second and third of the fractions $\frac{7}{15}$, $\frac{10}{21}$, $\frac{16}{35}$.

9. The sum of $\frac{2}{3}$ and $\frac{4}{15}$ is diminished by $\frac{1}{10}$; how often does the difference contain $\frac{3}{11}$ of the sum of $\frac{1}{8}$, $\frac{1}{9}$ and $\frac{1}{10}$?

10. Add together $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$, and subtract the sum from 2, multiply the difference by $\frac{3}{5}$ of $\frac{27}{40}$ of 88, and find what fraction the product is of 999.

11. From the sum of $3\frac{3}{4}$ and $4\frac{5}{6}$ subtract $6\frac{6}{7}$, multiply the difference by $2\frac{2}{5}$, and divide the product by $4\frac{1}{7}$.

12. Convert $\frac{1397}{2448}$ into a continued fraction, and find the first six converging fractions.

XI.

Reduce the following expressions to their simplest forms:—

1. $\left\{\frac{15}{6}-\frac{14}{15}+\frac{13}{14}-\frac{11}{12}\right\}\times\left\{\frac{7}{12}+\frac{5}{16}+\frac{8}{21}+\frac{11}{60}\right\}$

2. $81\,.\left\{\frac{3}{4}+\frac{5}{16}+\frac{7}{32}+\frac{9}{64}\right\}\times 64\left\{\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}\right\}$

3. $\left\{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right\}\div\left\{\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right\}-\left\{\frac{1}{3}-\frac{1}{5}\right\}\times\left\{\frac{1}{3}+\frac{1}{5}\right\}$

4. $\left\{\frac{11}{12}+\frac{13}{5}+\frac{7\frac{1}{2}}{15}-\frac{8\frac{3}{4}}{14}-\frac{16}{5}\right\}-\left\{\frac{5}{7}-\frac{1}{6}+\frac{2}{5}-\frac{3}{8}+\frac{9}{10}\right\}$

5. $\left\{\frac{816}{166463}-\frac{1}{1393}\right\}\div\left\{1+\frac{1}{1393}\times\frac{816}{166463}\right\}$

6. $\left\{\frac{7}{9}+\frac{9}{10}+\frac{10}{11}+\frac{11}{12}\right\}\div\left\{\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{7}{8}+\frac{8}{9}\right\}$

7. $\left\{\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right\}\times 17\left\{\frac{1}{35}+\frac{1}{36}\right\}-\frac{1}{12}\div\left\{\frac{11\frac{1}{2}}{15\frac{1}{3}}-\frac{1}{4}\right\}$

XII.

Prove the truth of the following equivalent expressions:

1. $1+\frac{1}{2}+\frac{1+3}{2+4}+\frac{1+3+5}{2+4+6}+\frac{1+3+5+7}{2+4+6+8}+\frac{1+3+5+7+9}{2+4+6+8+10}=4\frac{13}{20}$

2. $1+\frac{1}{2}+\frac{1\,.\,3}{2\,.\,4}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}+\frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8}+\frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2\,.\,4\,.\,6\,.\,8\,.\,10}=2\frac{181}{256}$

3. $\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}\right\}\div\left\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}\right\}=3\frac{1}{5}.$

4. $\frac{2}{7}\cdot\frac{7-2}{14}+\frac{4}{5}\cdot\frac{1}{10}+\frac{3}{5}\left\{\frac{1}{2}+\frac{11}{14}\right\}+\frac{3}{70}\left\{\frac{2}{7}+\frac{4}{5}\right\}=1$

5. $\left\{\frac{26}{333}-\frac{1}{23}-\frac{1}{29}\right\}\times\left\{\frac{14}{11}+\frac{8}{7}-\frac{1}{4}\right\}\times\left\{\frac{7}{10}+\frac{2}{13}\right\}=\frac{1}{4}\cdot\frac{1}{3.5.7.11}$

6. $\frac{\underline{|48}}{\underline{|13}\times\underline{|35}}\times 3\,.\,\frac{\underline{|48}}{\underline{|12}\times\underline{|36}}+3\,.\,\frac{\underline{|48}}{\underline{|11}\times\underline{|37}}+\frac{\underline{|48}}{\underline{|10}\times\underline{|38}}=\frac{\underline{|51}}{\underline{|13}\times\underline{|38}}.$[1]

7. $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\frac{8}{9}=\frac{1}{1}+\frac{1}{1}+\frac{1}{5}+\frac{1}{1}+\frac{1}{1}+\frac{1}{5}+\frac{1}{5}+\frac{1}{2}.$

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[1] The notation $\underline{|10}$ means $1\times 2\times 3\times 4\times 5\times 6\times 7\times 8\times 9\times 10$ or $1.2.3.4.5.6.7.8.9.10.$

CONCRETE FRACTIONS.

XIII.

1. How may any fractional parts of a given concrete number be found?

2. What is meant by reducing one concrete number to the fraction of another? Give examples.

3. Explain how a concrete fraction of one denomination is changed to an equivalent fraction of another denomination.

4. What is the value of $\frac{5}{8}$ of one guinea? and what fraction of one guinea is 13s. $1\frac{1}{2}$d.?

5. What is the value of $\frac{1}{2}$ of $\frac{7}{8}$ of $\frac{2}{3}$ of £12 12s. 6d.?

6. What part of £9 8s. 9d. is £8 13s. $0\frac{3}{4}$d.?

7. Express 3s. $8\frac{1}{4}$d. as the fraction of 1s. $6\frac{3}{4}$d.; and 1s. $6\frac{3}{4}$d. as the fraction of 3s. $8\frac{1}{4}$d.

8. What part of 2s. 6d. is equal to $\frac{3}{7}$ of 1s. $4\frac{1}{4}$d.?

9. What fraction of one pound is $\frac{5}{6}$ of $\frac{2}{3}$ of 18s. $6\frac{1}{2}$d.?

10. Which is the greater, $\frac{1}{21}$ of one pound or $\frac{1}{22}$ of one guinea?

11. What fraction of a pound is 4s. $9\frac{3}{5}$d.? How many shillings is the same fraction of a guinea?

12. Reduce 16s. $4\frac{2}{3}$d. to the fraction of a guinea.

13. Reduce $\frac{11}{15}$ of a pound to the fraction of a guinea; and conversely, $\frac{44}{63}$ of a guinea to the fraction of a pound.

14. If $\frac{5}{17}$ of an estate be worth £1,003 17s. 6d., what is the value of the whole estate?

XIV.

1. Explain the method of adding and subtracting concrete fractional numbers.

2. Add $\frac{3}{5}$ of 6s. 8d., $\frac{5}{7}$ of £2 3s. 9d., and $\frac{9}{11}$ of £4 14s. 5d.

3. Find the sum of $\frac{2}{3}$ of $\frac{5}{7}$ of £$7\frac{1}{2}$, and $\frac{5}{7}$ of £5 16s. $7\frac{1}{4}$d.

4. Add together £$100\frac{2}{3}$, $\frac{4}{17}$ of 100 guineas, and $\frac{2}{3}$ of $\frac{1}{11}$ of 1,000 crowns.

5. Add together $\frac{7}{8}$ of 50 guineas, $\frac{1}{2}$ of $\frac{2}{3}$ of $13\frac{1}{2}$ moidores, and $\frac{5}{7}$ of $11\frac{3}{4}$ crowns.

6. Express the sum of $\frac{5}{9}$ of a guinea, $\frac{3}{16}$ of a pound, and $\frac{7}{10}$ of a crown in the fraction of a guinea.

7. Add £$1\frac{7}{9}$, $9\frac{1}{8}$s., and $6\frac{4}{5}$d., expressing the sum in the fraction of £100.

8. Take $\frac{5}{39}$ of £6 6s. 9d. from $\frac{11}{51}$ of £10 16s. 9d.

9. Find the difference of $\frac{2}{9}$ of one guinea and $\frac{3}{8}$ of a pound in terms of the fraction of half a sovereign.

10. What fraction of £1 must be added to $\frac{5}{6}$ of $\frac{7}{8}$ of one guinea, in order that the sum may be one pound?

11. Express the difference of $\frac{7}{9}$ of £1 and $\frac{2}{5}$ of a guinea in the fraction of a crown.

12. What fraction of a crown is the difference between $\frac{3}{5}$ of $\frac{7}{8}$ of one guinea and $\frac{4}{5}$ of $\frac{2}{3}$ of £1?

13. A ship is worth £18,000, and a person who owns $\frac{7}{36}$ of it sells $\frac{3}{25}$ of his share. What is the value of his remaining share?

14. Out of £4 7s. 6d. one third is paid to *A* and one seventh to *B*; after this four elevenths of the remainder is paid to *A*, and the rest to *B*: find the sums respectively received by *A* and *B*.

XV.

1. Explain how the products and quotients of concrete fractions are to be interpreted.

2. Multiply $\frac{7}{11}$ of $\frac{4}{7}$ of £3 by $\frac{3}{5}$ of $2\frac{1}{2}$, and find its exact value.

3. Multiply £6 11s. $3\frac{7}{8}$d. by $13\frac{7}{8}$.

4. By what number must one ten-millionth of one pound be multiplied, in order that the product may be equal to one farthing?

5. Multiply 4s. $7\frac{2}{5}$d. by $3\frac{1}{2}$, and express the result in the fraction of a pound.

6. Divide £99 17s. $0\frac{1}{2}$d. by $19\frac{3}{4}$.

7. Divide $\frac{3}{5}$ of $\frac{7}{8}$ of 12s. by $\frac{2}{7}$ of $\frac{5}{6}$.

8. Divide £$7\frac{5}{7}$ by $15\frac{5}{8}$s. and state the nature of the quotient.

9. Divide the sum of $\frac{1}{9}$ of one pound and $\frac{5}{7}$ of one guinea by the difference of $\frac{3}{5}$ of a crown and $\frac{8}{13}$ of one shilling.

10. Add $\frac{1}{2}$ to $\frac{1}{4}$, subtract the sum from $\frac{5}{6}$, multiply this difference by $\frac{7}{8}$, divide the product by $\frac{9}{10}$, and supposing the original unity to have been one guinea, find the value of the resulting fraction in pounds, shillings, and pence.

11. Express £3,893 15s. 6d. and £3 17s. $11\frac{1}{2}$d. in the denomination of pounds, and find the quotient of the former divided by the latter, and state its nature.

12. In the new metric system of money, it has been proposed that the sixpence shall consist of 25 instead of 24 farthings. If a working man receive for his wages 17s. 6d. a week, show whether the change will be a gain or loss to him weekly, and how much in the year.

XVI.

1. Any sum of money can be expressed in pounds, twelfths of a pound, and a proper fraction of a twelfth; and five per cent. on the same may be immediately obtained by considering the pounds as shillings, the twelfths as pence, and the fraction of a twelfth as the same fraction of a penny. Explain the reason of this. Hence find 5 per cent. on £688 7s. 6d.

2. How much per cent. is 1s. 2d. in the pound?

3. What is the difference in the gain per cent. between selling goods at 2d. which cost 1½d., and selling goods at 2½d. which cost 2d. ?

4. What sum must *A* bequeath to *B* so that *B* may receive £1,000 after a legacy duty of £10 per cent. has been deducted?

5. If an article was bought for 3s. 6d. and sold for 3s. 10½d., what was the gain per cent. ? Find also the loss per cent. if the article was bought for 3s. 10½d. and sold for 3s. 6d.

6. A merchant sold goods for £75, and by doing so lost 10 per cent., whereas in the regular course of trade he should have gained 30 per cent. How much were they sold under their proper value?

7. If 7 per cent. be gained by selling goods for £69 11s., what percentage would be lost by selling them for £61 15s. ?

8. If an article be sold for 12s. 6d. there is a gain of 25 per cent.; what would be the loss or gain if it were sold for 10s. ?

9. By selling at 4½ per cent. profit a tradesman gains £47 14s.; what was the prime cost of his goods?

10. If a tradesman mark his goods on credit 20 per cent. above cash price, what ready money will he take for an article marked 26s. ?

11. If 10 per cent. be gained on the wholesale price of an article whose first cost is 5s. 6d., and 40 per cent. on the retail price, how much does the purchaser of such an article pay for profits to the wholesale and retail dealers?

12. The manufacturer of an article makes a profit of 10 per cent., the wholesale dealer a profit of 15 per cent., and the retailer makes a profit of 25 per cent. What is the cost to the manufacturer of an article which is retailed for 16 shillings?

XVII.

1. What fraction of a shilling, a pound, and a guinea is a franc?

2. Supposing the value of a franc to be 9½d., and that a commission of 3d. is charged on every sovereign exchanged; find the least number of sovereigns that can be changed with an exact number of francs.

3. If £1,000 be due from London to Paris when £1 is worth 25 francs, how much must be remitted when a pound is worth 27 francs?

4. If an Indian rupee be worth 2s. 4d., how many rupees are equal in value to £1,000 sterling?

5. If 10 scudi be worth 52½ francs, 16s. worth 20 francs, and 12 carlini worth 4s 2d., how many carlini are equivalent to 500 scudi?

6. If 12 carlini be worth 4s. 1d., and a Napoleon be worth 16s. how many carlini ought to be received for 15 Napoleons?

7. If 100 francs make £4, and 193 thalers make 19 guineas, how many thalers are there in 375 francs?

8. If 10 francs be worth 6 florins, and 15 florins be worth 4 moidores, how many francs must be given for 16 moidores?

9. The exchange between London and Paris is 25½ francs for one

pound sterling; between Paris and Amsterdam 117 francs for 55 florins; between Amsterdam and Hamburg 11 florins for 13 marks: required the exchange between London and Hamburg.

XVIII.

1. A bankrupt's debts amount to £4,586 5s., and his effects to £4,018 2s.; how much in the pound will his creditors receive, and what fractional part of the debt do they lose?

2. How much will a creditor lose on a debt of £5,342 5s. if he receive only 13s. 6d. in the pound?

3. A creditor receives 7s. 6d. in the pound of what was due to him, and thereby loses £75 5s. 3d. What was the sum due?

4. A bankrupt pays four dividends of 1s. 4d., 2s. 5¼d., 3s. 6½d., and 4s. 7¾d. respectively in the pound, and his creditors lose £965. What was the whole of his debt?

5. A creditor receives upon a debt of £272 a dividend of 11s. 6d. in the pound, and he receives a further dividend upon the deficiency of 3s. 9d. in the pound. What does the creditor receive on the whole debt?

6. A bankrupt's estate is worth £2,160. *A* has a claim for £1,200, *B* for £920, and *C* for £1,080. How much should each creditor receive?

7. A bankrupt owes three creditors £150, £200, and £350 respectively, and his whole property is worth only £275. How much will each creditor receive, and how much per cent. of their respective debts?

8. A house was to cost a builder £664, for which he was to receive £830. What would be his gain per cent.? But on becoming bankrupt the completion of the house cost £830, and was sold for £664. What was the loss per cent.? Explain why the rates per cent. differ.

9. *A, B, C* contribute respectively to an undertaking £105, £165, and £285, and gain £195. How shall they divide it equitably?

10. Three persons enter into a speculation and adventure £600, £800, and £1,000 respectively: they gain £1,000. Divide it among them so that they may have at the rate of 3, 4, and 5 per cent. respectively on the capital each advanced.

11. Three merchants contributed $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{6}$ respectively of the capital for a commercial adventure, on the condition that the gains should be divided at the rates of 7, 5, and 3 per cent. respectively on the sums each had contributed. If the gains were 1,500, what was the share of each?

12. Four merchants, *A, B, C, D*, trading with a capital of £36,000, find after a certain time their respective shares increased by £26 6s. 8d., £37 3s. 4d., £53 5s. 6d., and £63 4s. 6d. How much had they severally subscribed to the original capital?

XIX.

1. How many ounces, pennyweights, and grains are there in $\frac{49}{96}$ of a pound Troy? What fraction of a pound Troy is 6oz. 12dwts. 12grs.?

2. What is the value of a bar of gold weighing 87oz. 2dwts. 20grs. at £3 17s. $10\frac{1}{2}$d. per ounce?

3. Standard gold contains 22 parts of pure gold to two of alloy; if 20lb. Troy is coined into 934 sovereigns and one half-sovereign, find the weight of pure gold in a sovereign.

4. If a grain of pure gold be worth $2\frac{1}{2}$d., what should be the weight of a sovereign?

5. In standard gold how much per cent. is alloy?

6. If a sovereign by wear become one grain lighter than it ought to be, what loss will be sustained by selling 100 light sovereigns as bullion to the Bank of England at £3 17s. 9d. per ounce?

7. The English mint formerly coined $44\frac{1}{2}$ guineas out of one pound Troy of standard gold; it now makes $46\frac{29}{40}$ sovereigns out of the same quantity of gold. Is there any gain or loss by the change?

8. If an ounce of standard gold be worth £3 17s. $10\frac{1}{2}$d., what is the value of 10lb. of jewellers' gold in which the weight of alloy is 5 parts out of 12, the value of the alloy being neglected?

9. If 4oz. of gold, 17 carats fine, are mixed with 3oz., 13 carats fine, how much pure gold will there be in a gold ornament weighing $3\frac{1}{2}$oz. of the compound?

10. From 1797 to 1821 cash payments were suspended. Before that time the value of gold was £3 17s. $10\frac{1}{2}$d. per oz., but in 1815 it rose to £4 13s. 6d. per oz. How much per cent. had the currency depreciated?

11. How many sovereigns will weigh one ounce avoirdupois if 1869 weigh 40 pounds Troy?

12. In standard gold, if the alloy be silver at 5s. 6d. an ounce Troy, or copper at 1s. 6d. a pound avoirdupois, what is the difference in value of the sovereign when alloyed with copper and when alloyed with silver?

XX.

1. If 6 grains of silver are worth 5 farthings, what should be the weight of a crown?

2. Suppose the alloy in silver coin to be $\frac{1}{16}$ of the mass, what would a crown piece be worth if it were all silver?

3. A shilling weighs 3dwts. 15grs., of which 3 parts out of 40 are alloy, and the rest pure silver. What is the weight of alloy and of pure silver in a half-crown?

4. Supposing the alloy in a shilling to be $\frac{1}{13}$ of its mass, and the coin to be worth a farthing if it were all alloy, what would be its exact value if it were all pure silver?

5. If pure gold be worth £4 per ounce and pure silver 5s., what percentage of copper must be mixed with pure gold in order that the value of a given quantity of the compound metal may be 15 times that of an equal weight of a mixed metal containing 80 per cent. of pure silver?

6. The denarius of the times of the Roman republic weighed 60 grains of silver $\frac{29}{30}$ fine. Supposing a sestertium to be equivalent to 250 denarii, and standard silver which is $\frac{37}{40}$ parts fine to be worth 5s. 2d. per ounce, how much was a sestertium worth of our money?

7. A pound of silver is coined into 66 shillings, of which 62 only are issued. If nineteen half-crowns and fifteen sixpences are melted into bullion, and sent to the mint to be recoined, what sum will be reissued?

8. A mass of metal being composed of fine gold 15 parts, silver 4 parts, and copper 3 parts, find how much of each is required in making 18 cwt. of the composition.

9. The values of equal weights of gold and silver being £23 3s. 10d. and £1 13s. respectively, compare the weights of a sovereign and of a half-crown, if gold be valued at £3 17s. 10½d. per ounce.

10. If 66 shillings weigh one pound Troy, how many shillings are contained in one pound avoirdupois?

11. Allowing 44½ guineas to weigh one pound Troy, and 32 halfpence to weigh one pound avoirdupois, what is the difference in grains between the weights of a guinea and a halfpenny?

XXI.

1. Find the value of $\frac{25}{32}$ of one pound avoirdupois; and reduce 12oz. 8 drams to the fraction of a pound.

2. Reduce $\frac{7}{8}$ of $\frac{3}{4}$ of 15 cwt. to the fraction of $\frac{9}{10}$ of a ton.

3. How much ironstone ore must be raised from a mine, that on losing $\frac{17}{40}$ in roasting, and $\frac{8}{19}$ of the remainder in smelting, there may result 506 tons of pure metal?

4. If a shekel contained 240 grains, what was the weight in pounds avoirdupois of the head of Goliath's spear, which was 600 shekels of iron, and of his coat of mail, which consisted of 5,000 shekels of brass (1 Sam. xvii. 5, 7)?

5. Find a sum of money which shall be the same part of £61 9s. 1d. that 2cwt. 2qrs. 10lb. is of 36cwt. 1qr.

6. Show that $2\frac{1}{7}$ times the price of one hundred weight in pounds sterling is the price of one pound in pence. Ex. Let the price of one hundred weight be £14.

7. If a pound avoirdupois of copper is coined into 24 penny pieces, and a pound Troy of standard silver into 66 shillings, standard silver being a metal of which $\frac{37}{40}$ are pure silver and $\frac{3}{40}$ copper, find the value of one ounce of pure silver, if a pound avoirdupois contain 7,000 grains Troy.

8. If a mixed metal be composed of 11 parts of tin to 100 parts of copper, how much copper and how much tin will be required to make 24 cwt. of this metal?

9. Bronze contains 91 per cent. of copper, 6 of zinc, and 3 of tin. A mass of bell-metal (consisting of copper and tin only) and bronze fused together is found to contain 88 per cent. of copper, $4\frac{7}{8}$ of zinc, and $7\frac{1}{8}$ of tin. Find how much copper and how much tin are in bell-metal.

10. Sugar being composed of $49\frac{17}{20}$ per cent. of oxygen, $43\frac{53}{200}$ per cent. of carbon, and the remainder hydrogen, find how many pounds of each of these materials are contained in 112lb. of sugar.

11. Sulphate of lime consists of oxygen 32 parts, sulphur $16\frac{1}{10}$ parts, and calcium $20\frac{1}{2}$ parts; find the weight of each element in one pound of sulphate of lime.

12. In English gunpowder, 15 parts by weight are nitre, 2 parts sulphur, and 3 parts charcoal. How many pounds of each ingredient are required in making a ton of gunpowder?

XXII.

1. Multiply $7\frac{3}{5}$ lineal inches by $5\frac{7}{8}$, and by $5\frac{7}{8}$ lineal inches respectively.

2. Multiply $7\frac{3}{5}$ square inches by $5\frac{7}{8}$ lineal inches, and $5\frac{7}{8}$ square inches by $7\frac{3}{5}$ lineal inches.

3. Divide $7\frac{3}{5}$ square inches by $5\frac{7}{8}$, by $5\frac{7}{8}$ lineal inches, and by $5\frac{7}{8}$ square inches respectively.

4. Divide $7\frac{3}{5}$ cubic inches by $5\frac{7}{8}$, by $5\frac{7}{8}$ lineal inches, by $5\frac{7}{8}$ square inches, and by $5\frac{7}{8}$ cubic inches respectively.

5. Divide 3 lineal feet 8 inches by 1 lineal foot 10 inches.

6. Divide 8 square feet 48 inches by 3 lineal feet 4 inches.

7. Divide 3 cubic feet 192 inches by 18 square feet 48 inches.

8. Divide a cubic foot by 9, by 9 lineal inches, by 9 square inches and by 9 cubic inches respectively.

9. Explain if there be any difference between $30\frac{1}{4}$ square yards and $30\frac{1}{4}$ yards square; and if any, how much.

10. Multiply and divide 52 feet 6 inches by 5 feet 10 inches, and explain the results.

11. What length must be cut off a board which is $6\frac{3}{4}$ inches broad, so that the area may contain a square foot?

12. How many superficial feet of inch plank can be sawn out of a log of timber 20 feet 7 inches long, 1 foot 10 inches wide, and 1 foot 8 inches thick, supposing no waste in the sawing?

13. Of a field $\frac{1}{5}$ is meadow, $\frac{3}{8}$ is arable, and the remainder is 1 acre, 3 roods, 26 poles; find the quantities of meadow and arable land.

14. A piece of land of 200 acres is to be divided among 4 persons in proportion to their rentals from surrounding property. Sup-

posing these rents to be £500, £350, £800, and £160, how many acres must be allotted to each?

XXIII.

1. On measuring a distance of 32 yards with a rod of a certain length, it was found that the rod was contained 41 times with half an inch over. What is the distance approaching nearest to 77 yards which can be measured exactly by the same rod?

2. If $\frac{2}{3}$ of an inch on a map corresponds to 7 miles of a country, what distance on the map represents 20 miles?

3. Two scales 9 and 10 inches long respectively are each divided into 10 equal parts; the points of division being marked 1, 2, 3 . . . 10; if the scales are applied to one another so that the points marked 2 coincide, what is the distance between the points marked 9?

4. If 35 metres are equal to 39 yards, 17 metres to 9 toises, and 5 plethra to 124 toises, how many yards are there in 1,575 plethra?

5. The relative distances of three trees are 3, 4, and 5, and a rope 492 feet long just goes round them; find the respective distances.

6. If a yard measure be incorrect by one-eighth of an inch, find the error in measuring a distance of 500 yards 2 feet 6 inches. Explain, if there is any difference, whether the error be in excess or defect.

7. What fraction of the Earth's diameter is a mountain $5\frac{1}{2}$ miles high, if the radius of the Earth be 3,963 miles? Determine by what fractional parts of an inch could be represented (1) the highest point of Mont Blanc, which is about 15,700 feet above the level of the sea, on a globe of 16 inches in diameter; and (2) the highest peak of the Himalayas, Kunchinjunga, 28,178 feet above the level of the sea, on a globe of 30 inches diameter.

XXIV.

1. How many square yards, square feet, and square inches are there in a rectangle 34 yards 2 feet 11 inches long and 29 yards 1 foot 10 inches wide?

2. The length of one side of a rectangular field is 572 yards, and its area 50 acres, 2 roods, 32 poles; find the length of the other side.

3. How many yards of stuff 3 quarters broad will line a cloak $5\frac{1}{2}$ yards long and $1\frac{1}{4}$ yards broad?

4. What must be the width of a floor which is 17 feet 7 inches long, in order that it may contain the same area as another of 13 feet 4 inches by 22 feet 11 inches?

5. How many plots of ground containing $33\frac{3}{4}$ square yards can be cut off from a field containing 4 acres, 3 roods, 9 poles, $19\frac{1}{4}$ square yards, whose breadth is 135 yards; and what will be the width of the remaining strip, after the plots are marked off?

6. A piece of land whose length is 151 yards $1\frac{1}{4}$ feet, and breadth 35 yards, is to be exchanged for part of a strip of land of the same

quality whose breadth is 15 yards $2\frac{1}{2}$ feet. Find the length of the equivalent strip.

7. Find the whole surface of a room 22 feet 5 inches long, 18 feet 4 inches broad, and 11 feet 8 inches high.

8. The flooring of a room 14 feet 3 inches long, and 13 feet 4 inches broad, is composed of half-inch planks, each 8 inches wide and 10 feet long. What is the number of planks?

9. What length of carpet $\frac{3}{4}$ yard wide will cover a room whose length is 42 feet 5 inches, and breadth $31\frac{3}{4}$ feet?

10. If 30 yards of carpet $\frac{5}{8}$ yard wide cover the floor of a room, how many yards $\frac{3}{4}$ yard wide will answer the same purpose?

11. How many paving stones, each of them a foot long and $\frac{4}{15}$ of a foot wide, will be required for paving a street half a mile long and 45 feet wide?

12. Find the length of paper $\frac{7}{8}$ yard wide required to cover the walls of a room whose length is 27 feet 5 inches, breadth 14 feet 7 inches, and height 12 feet 10 inches.

XXV.

1. In a rectangular court which measures 96 feet by 84 feet there are four rectangular grass-plots, each measuring $22\frac{1}{2}$ feet by 18 feet. Find the cost of paving the remaining part of the court at $8\frac{1}{2}$d. per square yard.

2. What must be given per yard for carpet 27 inches wide, that the carpeting of a room 26 feet long and 15 feet 8 inches broad may cost £45?

3. The cost of carpeting a room whose length is 18 feet, at 3s. 6d. a square yard, is £5 12s.; and the cost of painting the walls, at 4s. 6d. a square yard, is £17. Find the height and breadth.

4. The expense of carpeting a room 18 feet long was £$7\frac{1}{5}$, but if the breadth of the room had been 4 feet less than it was, the expense would have been £$5\frac{2}{5}$. What was the breadth of the room?

5. What is the cost of papering the walls of a room 12 feet 4 inches high, 16 feet 6 inches long, and 14 feet 3 inches wide, the paper being 2 feet 6 inches wide, and costing 3s. 6d. a yard, the workman also charging a farthing for every square foot?

6. If the painting of the walls of a room 20 feet 3 inches long and 4 feet 8 inches broad cost £2 3s. $7\frac{3}{4}$d. at $7\frac{1}{2}$d. per square yard, what is the height of the room?

7. A room is 14 feet 6 inches high, 20 feet wide, and 22 feet long. What will be the cost to cover it with paper 2 feet 6 inches wide, whose price is $10\frac{1}{4}$d. a yard? Allow 8 feet by 5 feet 3 inches for each of two doors, 6 feet 6 inches by 6 feet for a fireplace, and 12 feet by 5 feet 7 inches for one window.

XXVI

1. What is the solid content of a cube whose edge is 6 feet 6 inches?

2. How many cubes of $2\frac{1}{4}$ inches in the edge can be cut from a cube of which the edge is 27 inches?

3. How many cubes whose edges are $\frac{3}{4}$ inch long can be contained in a box of which the base is 18 square inches, and the depth $7\frac{1}{2}$ inches?

4. What length must that solid be to contain a cubic foot which is 8 inches broad and 3 inches wide?

5. If a solid parallelopiped contain 100 cubic yards, and its base contain 100 square feet, find its height.

6. How many cubic yards of earth were dug from an excavation a mile and a half long, 30 yards wide, and 24 feet deep?

7. What is the number of cubic feet and inches in a piece of masonry 9 feet 3 inches long, 11 feet 5 inches high, and 3 feet 2 inches thick?

8. Find the length of a solid whose thickness is 1 foot, breadth 18 inches, and solid contents 3 cubic feet 216 inches.

9. What must be the length of a trench which is $5\frac{1}{2}$ feet deep and $9\frac{1}{3}$ feet wide, in order that the entire excavation may contain 7,040 cubic feet?

10. A rectangular cistern whose length is $13\frac{3}{8}$ feet, and breadth 6 feet, contains $294\frac{1}{4}$ cubic feet. What is the depth?

11. How many bricks will be required to build a wall 20 yards long, $7\frac{1}{2}$ feet high, and 13 inches thick, supposing a brick to be 9 inches long, 4 wide, and 3 thick? If there be in the wall a doorway 6 feet 4 inches high and 4 feet wide, how many bricks less would be required?

12. A chest is 6 feet 2 inches long, 3 feet 4 inches wide, and 2 feet 9 inches deep, and is composed of boards $1\frac{1}{2}$ inch thick. Find the quantity of wood used, the contents of the chest, and the difference between its external and internal surfaces.

XXVII.

1. If a cubic foot of wood weigh 12lb., what is the weight of a beam whose length, width, and depth are 24, $2\frac{3}{4}$, and $2\frac{1}{2}$ feet?

2. The weight of a cubic foot of Portland stone is 156 pounds; find the weight of the first stone of the Fitzwilliam Museum, at Cambridge, the mean dimensions of it being: length 7 feet, breadth 3 feet 9 inches, and thickness 2 feet.

3. What is the weight of a block of stone 12 feet 6 inches long, 6 feet 6 inches broad, and 8 feet 3 inches thick, when a block of the same stone 5 feet long, 3 feet 9 inches broad, and 2 feet 6 inches thick, weighs 7,500lb.?

4. Find the weight of a rectangular vessel of iron an inch and a half thick without a top, the vessel being 10 feet 8 inches long, 8 feet 4 inches broad, and 5 feet deep, supposing a pound weight of iron to have a bulk of 4 cubic inches.

5. How many cubic feet of lead one-eighth of an inch thick would be required to line a cistern whose length, breadth, and depth are 7 feet 10 inches, 6 feet 3 inches, 3 feet 4 inches?

6. A cubic foot of copper weighs 4cwt. 3qrs. 24lb. 4oz., and can be drawn into a wire 1 mile 3 furlongs 5 poles long. Find the weight of copper requisite for a single wire of 50 miles. Find also the area of a section of the wire.

7. How long is an iron bar that contains a cubic foot of iron when its breadth is $\frac{3}{4}$ of an inch, and thickness is $\frac{1}{2}$ an inch?

8. Find how many square yards a cubic foot of gold would cover when beaten out $\frac{19}{100000}$ of an inch thick, and show that its extent is greater than $\frac{1}{442}$, but less than $\frac{1}{441}$ of a square mile.

9. If gold can be beaten out so thin that a grain will form a leaf of 56 square inches, how many of these leaves will make an inch thick, the weight of a cubic foot of gold being 10cwt. 95lb. Avoirdupois?

XXVIII.

1. If 6 men earn £7 6s. 3d. in $7\frac{1}{2}$ days, how much will 10 men earn in $11\frac{3}{4}$ days?

2. If 4 men earn £4 7s. 6d. by working 5 days of 10 hours each, how much would 7 men earn in 3 weeks working 8 hours a day?

3. If 7 men and 10 boys earn £6 a week, and 3 men and 5 boys earn £11 in 4 weeks, in what time will 5 men and 4 boys earn £14?

4. The wages of *A* and *B* together for $7\frac{1}{2}$ days amount to the same sum as the wages of *A* alone for $12\frac{6}{7}$ days. For how many days will the sum pay the wages of *B* alone?

5. If 3 men do as much work as 5 women, and 2 women as much as 5 children; divide between 9 men, 11 women, and 6 children a day's wages amounting altogether to £3 5s. 1d.

6. The wages of one man, one woman, and three children amount to 30s. per week. The man and one child earn together 4 times as much as the woman. The man and three children earn 5 times as much as the woman. Find the wages of each.

7. If 20 men, 40 women, and 50 children receive among them £350 for 7 weeks' work, and 2 men receive as much as 3 women or 5 children, what sum does a man, a woman, and a child receive weekly?

8. Three men are employed on a work, working respectively 8, 9, 10 hours a day, and receiving the same daily wages. After 3 days each works one hour a day more, and the work is finished in 3 days more. If the total sum paid for wages be £2 7s. $6\frac{1}{4}$d., how much of it should each receive?

XXIX.

1. If 100 men can perform a piece of work in 30 days, how many men can perform another piece of work thrice as large in one-fourth part of the time?

2. If 5 men or 7 women can do a piece of work in 37 days, how long will a piece of work twice as great occupy 7 men and 5 women?

3. If 12 men and 9 boys do a piece of work in 25 days, find how many men, assisted by an equal number of boys, would be required to do the same in 9 days, supposing 2 men to do as much as 3 boys.

4. If 3 men, 4 women, 5 boys, or 6 girls can perform a piece of work in 60 days, how long will it take 1 man, 2 women, 3 boys, and 4 girls, working together?

5. Men, women, and children are employed to do a piece of work. A man can do twice as much as a woman, and 6 times that a child; 8 women and 18 children can do the work in 40 days. How long would it take 10 men to do it?

6. Three men and two women take three days to do a piece of work which four women and three boys can do in 5 days, and one man does as much as three women in the same time. How long would it take a man, a woman, or a boy to do the work alone?

7. If 5 men and 2 boys can complete a piece of work in 6 days which 5 boys and 2 men will complete in 12 days, how long would a single man or boy take to complete it by himself?

8. If a man can do treble, and a woman double the work of a boy in the same time, how long would 9 men, 15 women, and 18 boys take to do double the work which 7 men, 12 women, and 9 boys complete in 250 days?

9. If 6 men and 2 boys can reap 13 acres in 2 days, and 7 men and 5 boys can reap 33 acres in 4 days, how long will it take 2 men and 2 boys to reap 10 acres?

XXX.

1. Two persons, A and B, finish a work in 20 days, which B by himself could do in 50 days. In what time could A finish it by himself? How much more of the work is done by A than by B?

2. A alone can do a piece of work in 11 days, and B alone can do it in 17 days. How long would they take to do it together?

3. A and B do a piece of work together in 6 days, and B can do a fifth of the same in a day and a half. How long would each be in doing it alone?

4. A and B can build a wall together in 12 days, A and C in 15, B and C in 20. In how many days can they do it (1) working all together; (2) each working separately?

5. A and B can do a piece of work alone in 15 and 18 days respectively; they work together at it for 3 days, when B leaves, but A continues, and after 3 days is joined by C, and they finish it together in 4 days. In what time would C do the work by himself?

6. A and B can perform a work in 7 days, A and C in 8 days, and A, B, and C in 5 days. In what time can B and C perform it?

7. A does $\frac{2}{3}$ of a piece of work in 4 hours, B does $\frac{3}{4}$ of what remains

in 1 hour, and *C* finishes it in 20 minutes. How long would they have been in doing the whole if they had worked together?

8. If *A* can perform a work in 9 days, *B* in 10, and *C* in 11, in what times could *A* and *B*, *A* and *C*, and *B* and *C* respectively complete the work?

9. *A*, *B*, and *C* can each complete a certain work in 9, 12, and 18 days respectively. If *A* and *B* work together for 2 days, and *A* and *C* afterwards for 2 more, how long will *B* and *C* working together take to finish the work?

10. If *A* can do as much work in 5 hours as *B* can do in 6 hours, or as *C* can do in 9 hours, how long will it take *A* to complete a piece of work, one half of which has been done by *B* working 12 hours, and *C* working 24 hours?

11. If *A* in 2 days can do as much work as *C* in 3 days, and *B* in 5 days as much as *C* in 4 days, how long will *B* require to execute a piece of work which *A* can accomplish in 6 weeks?

12. *A* is twice, and *B* is just one and a half times as good a workman as *C*. The three work together for two days, and then *A* works on alone for half a day, and *B* for one day. How long would it have taken *A* and *C* together to complete as much as the three will have thus performed?

13. If *A* can by himself perform a certain quantity of work in 5 days, *B* twice as much in 7, and *C* 4 times as much in 11 days, in what time can *A*, *B*, and *C* together perform 3 times the original quantity?

14. *A* can do as much work in 2 days as *B* in 3 days, and *B* as much in 4 days as *C* in 5 days. In what time could *A*, *B*, and *C* together do a piece of work which *A* can do in 11 days?

XXXI.

1. If a cubic foot of water weigh 1,000 ounces avoirdupois, find the number of grains Troy in a cubic inch of water.

2. A rectangular tank is 13 feet 6 inches long by 9 feet 9 inches wide. How many cubic yards of water must be drawn off to make the surface sink 1 foot?

3. How many tons of water will a cistern contain whose length, breadth, and depth are respectively 18 feet 8 inches, 10 feet 4 inches, and 6 feet 9 inches? What is the amount of pressure of the water on the bottom of the cistern?

4. If 13 cubic feet make 81 gallons, how many hogsheads will be contained in a cistern 42 feet long, 39 feet broad, and $10\frac{1}{2}$ feet deep?

5. A cistern is $\frac{2}{3}$ full of water, and after 35 gallons are drawn off it is found to be $\frac{3}{8}$ full. How many gallons does it contain?

6. Two equal wine-glasses are filled with a mixture of spirit and water, one containing 3 parts of water and one part of spirit, and the

other 4 parts of water and 3 parts of spirit. When the contents of the two glasses are mixed in a tumbler, find how many parts of the whole mixture are wine and water.

7. Four gallons of wine and one of water are poured into a vessel, and one gallon of the mixture is poured out into a second vessel; then one gallon of water is poured into the first vessel, and the process is performed four times. Determine the quantity of wine in each vessel.

8. If a cubic foot of atmospheric air weigh an ounce and a quarter under the ordinary pressure, what would be the weight of air contained in a room 30½ feet long, 20⅓ wide, and 10¼ high?

9. If the pressure of the atmosphere on a square inch at the surface of the earth be about 14 pounds 11 ounces avoirdupois, what is the pressure on a square foot and on a square yard?

10. What is the pressure on a square foot at the bottom of a pond 20 feet deep, if the weight of a cubic foot of water be 1,000 ounces? If the pressure of the atmosphere be taken into account, what is the total pressure?

11. If the weight of a cubic foot of water be 1,000 ounces avoirdupois, and mercury be 13½ times as heavy as water, find the height of a column of water which shall be equal to the pressure of mercury in the barometer when it stands at 30½ inches.

XXXII.

1. If a vessel of water be emptied in 5 hours through a spout *A*, and in 3 hours through a spout *B*, in what time will it be emptied through both spouts together?

2. If a cistern when full of water can be emptied in 15 minutes by a pipe, and when empty can be filled by another in 20 minutes; if the cistern be full, in what time can it be emptied by both pipes being opened at the same time?

3. If a pipe which conveys 1⅗ gallons in a minute, lower the surface of water in a tank 1½ inches in 45 minutes, in what time will a pipe which conveys 4 gallons in a minute lower the surface 5 inches?

4. A cistern is filled by two spouts in 20 minutes and in 24 minutes respectively, and emptied by a tap in 30 minutes; if the cistern be empty, what portion of the cistern will be filled in 15 minutes, when all three are opened together?

5. A cistern has 3 pipes, *A*, *B*, *C*; *A* will fill it in 3 hours, *B* will fill it in 4, and *C* will empty it in 1 hour. The cistern being empty, these pipes are opened at 1, 2, and 3 o'clock respectively. At what time will the cistern be full or empty, and which?

6. In what part of a day will four fountains, being opened together, fill a cistern, which if severally opened, they would each fill it in one day, half a day, the third and the sixth part respectively?—(*Lilavati*).

7. Translate and solve the problem:—

Χαλκεός ἔιμι λέων, κρούνοι δὲ μοι ὄμματα δοιὰ,
  Καὶ στόμα, καὶ τὸ θέναρ δεξίτεροῖο πόδος.
Πλήθει δὲ κρητῆρα δυ' ἤμασι δεξίον ὄμμα,
  Καὶ λαιὸν τρισσο͂ις, καὶ πισύρεσσι θέναρ:
Ἄρκιον ἕξ ὣραις πλῆσαι στόμα, ἔι δ'ἅμα πάντα,
  Καὶ στόμα, καὶ γλῆναι καὶ θέναρ, ἐιπὲ πόσοις.

XXXIII.

1. If one person take steps of 29 inches each and another person steps of 31 inches; if the latter take 59, and the former 61 in a minute; what is the difference of time between each person walking a mile?

2. *A* and *B* walk to meet each other from two places 100 miles distant. *A* walks 6 miles an hour and *B* 4 miles an hour. At what point on the road do they meet, and at what two times are they 50 miles apart from each other?

3. How long will a column of 10,000 men, 4 deep, require to march through a defile of 5 miles at the rate of 75 paces of 2 feet each in one minute, supposing each rank of 4 men to occupy 20 inches in depth?

4. If two trains be moving in contrary directions at the rate of 30 miles an hour, and each of them be 88 yards long, how long will they take to pass each other? But if one of the trains 88 yards long was moving 30 miles an hour, and the other 120 yards long was moving 40 miles an hour, find the time in which they would pass one another.

5. Two trains 100 yards and 150 yards long respectively, are moving in the same direction at the rates of 30 and 40 miles an hour. In what time will one pass the other?

6. If sound travel through air at the rate of 1,130 feet per second, through water at 4,700 feet per second, and through land at 7,000 feet per second, in what times could sound be transmitted a distance of 6 miles through each of these media?

7. A person saw the flash of a gun fired from a ship at sea distant a mile and 480 yards, and 5 seconds afterwards saw the flash of another gun fired from another ship in a line between the first ship and himself, and two seconds still later heard two reports simultaneously. Find the distance between the ships.

8. A person shooting at a target at a distance of 500 yards hears the bullet strike the target 4 seconds after he fired. A spectator equally distant from the target and the shooting station hears the shot strike $2\frac{1}{2}$ seconds after he heard the report. Find the velocity of sound.

9. If the velocity of electricity be 288,000 miles per second, what time would it take to travel round the earth, whose circumference is about 24,900 miles?

10. While Roemer was engaged in observing the eclipses of Jupiter's satellites, he found that when Jupiter was in opposition the eclipses happened $8' 13''$ earlier than they should according to the astronomical tables, and when Jupiter was in conjunction these eclipses

happened 8′ 13″ later. Find the velocity of light if the radius of the earth's orbit be 93,000,000 miles.

11. Two cogged wheels, of which one has 16 cogs and the other 30, work in each other. If the first wheel turn 18 times in 10 seconds, how many times will the other turn in 25 seconds?

12. If two bodies move in the circumference of a circle, the swifter making a revolution in 5 hours and the slower in 9, supposing they start from the some point, when will one overtake the other?

13. The periodic times of four bodies being 24, 22, 20, and 18 days respectively, in what times after leaving a conjunction will they all be again in conjunction, and what number of revolutions will each have performed?

XXXIV.

1. A clock is set at 12 o'clock on Saturday night, and at noon on Tuesday it is 3 minutes too fast. Supposing its rate regular, what will be the true time when the clock strikes 4 on Thursday afternoon?

2. Find the different times at which the hour and minute hand of a clock are in conjunction, in opposition, and at right angles to one another, between noon and midnight. If the hands were alike, at what times of the day might they be mistaken the one for the other?

3. The seconds hand of a watch revolves about the same axis as the hour and minute hands. Determine all the positions in which the three hands are together, in the same straight line, and at right angles, during one revolution of the hour hand.

4. If a watch be 4 min. 8 $\frac{9}{32}$ sec. too slow at 9 hrs. 30 min. a.m. on Tuesday, and loses 2 min. 45 sec. daily, what will be the time indicated at 5 hrs. 15 min. p.m. on the following Friday?

5. A watch which is 10 minutes too fast at noon on Monday loses 3 min. 10 sec. daily. What will be the time indicated by the watch at a quarter past 10 on the morning of the following Saturday?

6. What is the magnitude of the angle between the hour and minute hand of a clock at $\frac{1}{4}$ past 11?

7. It is between 2 and 3 o'clock, and the hands of the clock are equally inclined to the vertical on opposite sides. In what time will they be inclined to the vertical again on opposite sides?

8. A watch set accurately at 12 o'clock indicates 10 minutes to 5 at 5 o'clock. What is the exact time when the watch indicates 5 o'clock? If it indicated 10 minutes past 5 at 5 o'clock, what would be the exact time when the hands indicated 5 o'clock?

9. One clock gains 2 minutes in 3 days, and another loses 6 minutes in 7 days; if they were set right at 12 noon to-day, when will their times differ by a quarter of an hour?

10. Two clocks begin to strike 12 together; one strikes in 35 seconds the other in 25. What fraction of a minute is there between their seventh strokes?

RESULTS, HINTS, ETC., FOR THE EXERCISES.

ABSTRACT FRACTIONS.

I.

1. Art. 1. 2. Arts. 1, 9. 3. The improper fractions are given. 4. $\frac{12+9+6}{3}=4+3+2=9$, $\frac{12\times9\times6}{3}=4\times9\times6=216$, and $9=\frac{1}{24}$ of 216. 5. Art. 11 and note. 6. Art. 12. 7. Art. 13, note. 8. Art. 15 and note. 9. Art. 10 and note. 10. Arts. 15, 16, and note. 11. Art. 16.

12. The numbers which are aliquot parts of 12 are 1, 2, 3, 4, 6. Thus 1 is $\frac{1}{12}$ of 12, 2 is $\frac{1}{6}$ of 12, 3 is $\frac{1}{4}$ of 12, 4 is $\frac{1}{3}$ of 12, 6 is $\frac{1}{2}$ of 12. The other numbers less than 12, namely, 5, 7, 8, 9, 10, 11, can be divided into two or more aliquot parts.

Thus, $\frac{5}{12}=\frac{4}{12}+\frac{1}{12}=\frac{1}{3}+\frac{1}{4}$ of $\frac{1}{3}$. | $\frac{9}{12}=\frac{6}{12}+\frac{3}{12}=\frac{1}{2}+\frac{1}{2}$ of $\frac{1}{2}$.
$\frac{7}{12}=\frac{6}{12}+\frac{1}{12}=\frac{1}{2}+\frac{1}{6}$ of $\frac{1}{2}$. | $\frac{10}{12}=\frac{6}{12}+\frac{4}{12}=\frac{1}{2}+\frac{1}{3}$.
$\frac{8}{12}=\frac{6}{12}+\frac{2}{12}=\frac{1}{2}+\frac{1}{3}$ of $\frac{1}{2}$. | $\frac{11}{12}=\frac{6}{12}+\frac{3}{12}+\frac{2}{12}=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$.

It may also be noted that the aliquot parts of 20 and of 100 are much employed in calculations connected with mercantile transactions.

II.

The following are the fractions in the lowest terms found by dividing the numerator and denominator of each fraction by their greatest common measure respectively:—

$\frac{11}{20}$, $\frac{55}{64}$, $\frac{4}{59}$, $\frac{55}{8}$, $\frac{1309}{360}$, $\frac{60}{77}$, $\frac{2993}{3321}$, $\frac{23}{33}$, $\frac{19}{34}$, $\frac{131}{191}$, $\frac{276}{397}$, $\frac{8}{9}$, $\frac{7}{9}$, $\frac{13}{14}$, $\frac{61}{211}$, $\frac{10996}{34661}$, $\frac{11761}{122387}$.

III.

The following are the fractions as required:—

1. $\frac{5}{15}$ and $\frac{3}{15}$: $\frac{8}{12}$ and $\frac{9}{12}$: $\frac{35}{105}$, $\frac{20}{105}$ and $\frac{12}{105}$: $\frac{168}{420}$, $\frac{240}{420}$ and $\frac{175}{420}$: $\frac{4680}{6552}$, $\frac{2457}{6552}$, $\frac{2912}{6552}$ and $\frac{5544}{6552}$: $\frac{2376}{3960}$, $\frac{2475}{3960}$, $\frac{1760}{3960}$ and $\frac{2880}{3960}$.

2. $\frac{6}{9}$ and $\frac{4}{9}$: $\frac{8}{18}$, $\frac{12}{18}$ and $\frac{7}{18}$: $\frac{20}{36}$, $\frac{18}{36}$, $\frac{3}{36}$ and $\frac{5}{36}$: $\frac{40}{100}$, $\frac{75}{100}$, $\frac{35}{100}$ and $\frac{51}{100}$: $\frac{160}{1000}$, $\frac{375}{1000}$, $\frac{175}{1000}$, $\frac{16}{1000}$ and $\frac{3}{1000}$.

3. $\frac{15}{100}$ and $\frac{28}{100}$: $\frac{30}{600}$, $\frac{24}{600}$ and $\frac{25}{600}$: $\frac{45}{72}$, $\frac{42}{72}$ and $\frac{44}{72}$: $\frac{105}{168}$, $\frac{98}{168}$, $\frac{24}{168}$ and $\frac{56}{168}$: $\frac{120}{240}$, $\frac{100}{240}$, $\frac{32}{240}$, $\frac{210}{240}$ and $\frac{75}{240}$.

IV.

1. $\frac{4}{5}$. 2. $1\frac{6}{35}$. 3. $7\frac{20}{21}$. 4. $55\frac{73}{120}$. 5. $1\frac{212}{459}$. 6. $7\frac{1}{12}$. 7. $2\frac{1}{8}$. 8. $5\frac{10}{21}$. 9. $1\frac{547}{910}$. 10. $465\frac{121}{315}$. 11. $1\frac{11}{72}$. 12. $4\frac{731}{3080}$. 13. $305\frac{41}{42}$. 14. $3536\frac{5}{12}$. 15. $33\frac{212}{385}$.

16. $8\frac{399}{616}$. 17. $5\frac{373}{1260}$. 18. 1. 19. $251\frac{3}{4}$. 20. $\frac{31}{44}$. 21. $1\frac{7}{19}$. 22. $\frac{1303151}{166463\times1393}$.
23. $43\frac{23}{126}$.

Note.—The four fractions in example 18 may be more readily added together by resolving the denominators of the fractions into their prime factors.

$$\text{Here } \frac{7}{2431}+\frac{29}{2717}=\frac{1}{11\times13}\left\{\frac{7}{17}+\frac{29}{19}\right\}=\frac{626}{11\times13\times17\times19}.$$

$$\text{And } \frac{589}{3553}+\frac{3446}{4199}=\frac{1}{17\times19}\left\{\frac{589}{11}+\frac{3146}{13}\right\}=\frac{45563}{11\times13\times17\times19}.$$

$$\text{Sum of the four fractions}=\frac{46189}{11\times13\times17\times19}=\frac{46189}{46189}=1.$$

V.

1. $\frac{11}{24}$. 2. $2\frac{1}{30}$. 3. $1\frac{29}{30}$. 4. $10\frac{1}{28}$. 5. $12\frac{3}{5}$. 6. 122. 7. $\frac{29}{108}$. 8. $238\frac{65}{66}$. 9. $\frac{11}{108}$. 10. $7\frac{3}{11}$. 11. $42\frac{41}{175}$. 12. $345\frac{73}{96}$. 13. $21\frac{7}{24}$. 14. $234\frac{76}{117}$. 15. $1959\frac{217}{396}$.

VI.

1. $\frac{1}{6}$. 2. $\frac{1}{12}$. 3. $\frac{1}{2}$. 4. $\frac{5}{7}$. 5. $\frac{15}{77}$. 6. $3\frac{1}{2}$. 7. 252. 8. $\frac{1}{168}$. 9. $1713379\frac{1}{27}$. 10. $\frac{5}{24}$. 11. $\frac{1}{3060}$. 12. $\frac{5}{84}$. 13. $4\frac{47}{112}$. 14. 1. 15. $4\frac{16}{21}$. 16. $193\frac{11}{30}$. 17. $3460\frac{1}{20}$. 18. $115\frac{47}{80}$. 19. $\frac{1}{21}$. 20. $\frac{3}{425}$. 21. $1783\frac{3}{5}$. 22. $\frac{15037957}{18662400}$. 23. $100002\frac{1}{1000000}$. 24. $90909090\frac{10}{11}$.

VII.

1. $\frac{10}{21}$. 2. $3\frac{3}{7}$. 3. $\frac{5}{6}$. 4. $1\frac{1}{9}$. 5. $\frac{3}{32}$. 6. $1\frac{24}{25}$. 7. $2\frac{3}{16}$. 8. $1\frac{5}{9}$. 9. 1. 10. $2\frac{3}{16}$. 11. $2\frac{16021}{43044}$. 12. $8\frac{317}{720}$.

VIII.

1. Art. 12, note, $\frac{18}{29}$ is greater than $\frac{23}{38}$, and $\frac{23}{38}$ is greater than $\frac{11}{19}$.

2. $\frac{253}{23}$, $\frac{206}{23}$, $\frac{391}{23}$. 3. $\frac{1}{18}$th part.

4. Their relative magnitudes will be obvious when the fractions are reduced to the same denominator. 5. Art. 7. 6. Art. 12, note.

7. If 3 be added to the numerator and denominator of the proper fraction $\frac{7}{12}$

It becomes $\dfrac{7+3}{12+3}$ or $\dfrac{10}{15}$ or $\dfrac{2}{3}$,

And $\frac{2}{3}$ is greater or less than $\frac{7}{12}$

According as $\frac{8}{12}$ is greater or less than $\frac{7}{12}$. Art. 12.

But $\frac{8}{12}$ is obviously greater than $\frac{7}{12}$.

It follows that $\dfrac{7+3}{12+3}$ is greater than $\dfrac{7}{12}$,

or that the proper fraction $\frac{7}{12}$ is increased by adding the same number 3 to its numerator and denominator.

In a similar way it may be shewn that the proper fraction $\frac{7}{12}$ is diminished by subtracting the same number 3 from its numerator and denominator.

The remaining examples 8, 9, 10, 11, 12, offer no difficulties.

With respect to Ex. 12 see Art. 7.

IX.

1. Reduce to single fractions and find the sum. 2. The result is $484\frac{899}{905}$. 3. $\frac{11}{120}$. 4. $\frac{13}{4145}$. 5. The sum of the fractions is $\frac{3}{11}$: and the lowest fraction required with denominator 1000, will be next greater than $\frac{8}{11}$, which will be found to be $\frac{728}{1000}$. 7. $\frac{13}{28}$. 8. $6\frac{23}{24}$. 9. $\frac{18}{13}$ excess, $\frac{18}{91}$ of 7. 10. 300. 11. $\frac{17}{200}$ can be subtracted 11 times from $\frac{197}{200}$, and leaves a remainder $\frac{10}{200}$. 12. $3+5+7=15$, and $\dfrac{3}{15}+\dfrac{5}{15}+\dfrac{7}{15}=1$.

X.

1. $\frac{21}{8}=9\times\frac{7}{24}$. 2. The difference is $\frac{41}{420}$: the product 1. 3. $\frac{5}{4}$. 4. $10\frac{4}{15}$. 5. $\frac{1485839}{5224032}$. 6. $\frac{11}{18}$. 7. The quotient is greater than the product. 8. $\frac{1}{98}$. 9. $99\frac{1}{11}$. 10. $\frac{473}{18500}$. 11. 1.

12. $\frac{1397}{2448}=\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1+}\frac{1}{3+}\frac{1}{3+}\frac{1}{13}$: 1, $\frac{1}{3}$, $\frac{4}{7}$, $\frac{5}{9}$, $\frac{9}{16}$, $\frac{32}{57}$.

XI.

1. $2\frac{12373}{44100}$. 2. 11830. 3. $1\frac{209}{225}$. 4. $2\frac{467}{840}$. 5. $\frac{1}{239}$. 6. $\frac{1404}{1913}$. 7. $\frac{907}{1800}$.

XII.

The operations indicated in the first four expressions offer no difficulty.

5. The expression is equivalent to $\frac{2.13}{3.3.37.23.29}\times\frac{23.29}{4.11.7}\times\frac{3.37}{2.5.13}$.

6. The left side of the expression can be put under the form—

$$\frac{\underline{|48}}{\underline{|10}.\underline{|35}}\left\{\frac{1}{11.12.13}+\frac{3}{11.12.36}+\frac{3}{11.36.37}+\frac{1}{36.37.38}\right\}$$

And $\frac{1}{11.12.13}+\frac{3}{11.12.36}=\frac{1}{11.12}\left\{\frac{1}{13}+\frac{1}{12}\right\}=\frac{25}{11.12.13.12}$.

$$\frac{25}{11.12.13.12}+\frac{3}{11.36.37}=\frac{1}{11.12}\left\{\frac{25}{13.12}+\frac{1}{37}\right\}=\frac{1081}{11.12.13.12.37}.$$

$$\frac{1081}{11.12.13.12.37}+\frac{1}{36.37.38}=\frac{1}{36.37}\left\{\frac{3243}{11.12.13}+\frac{1}{38}\right\}=\frac{124950}{11.12.13.36.37.38}$$

$$=\frac{49.50.51}{11.12.13.36.37.38}.$$

Therefore $\frac{\underline{|48}}{\underline{|10}.\underline{|35}}.\frac{49.50.51}{11.12.13.36.37.38}=\frac{\underline{|51}}{\underline{|13}.\underline{|38}}$.

7. See Art. 11, note.

Concrete Fractions.

XIII.

1, 2, 3. Art. 17, and note. 4. The direct and converse. 5. £2 10s. $5\frac{23}{32}$d. 6. $\frac{2769}{3020}$. 7. $\frac{59}{15}$ and $\frac{15}{59}$. 8. $\frac{13}{56}$. 9. $\frac{445}{864}$. 10. $\frac{1}{22}$ of 1 guinea. 11. £ $\frac{6}{5}$, and $3\frac{9}{25}$s. 12. $\frac{295}{378}$. 13. The direct and converse. 14. $\frac{5}{17}$ of estate=£1003 17s. 6d.: $\frac{1}{17}$=£200 15s. 6d., and $\frac{17}{17}$ or whole estate=£3413 3s. 6d.

XIV.

1. Art. 18. 2. £5 12s. $4\frac{3}{5}$d. 3. £7 14s. $5\frac{3}{4}$d. 4. £140 10s. $11\frac{145}{187}$d. 5. £54 2s. $2\frac{4}{7}$d. 6. $\frac{227}{252}$ of 1 guinea. 7. $\frac{16289}{720000}$ of £100. 8. £1 10s. 6d. 9. $\frac{17}{60}$ of half a sovereign. 10. $\frac{15}{64}$ of £1. 11. $\frac{322}{225}$ of a crown. 12. $\frac{43}{600}$ of a crown. 13. Remaining share $\frac{77}{450}$, and its value £3080. A received £2 5s. 10d., and B £2 1s. 8d.

XV.

1. See on Multiplication and Division of Concrete integers. 2. $\frac{18}{11}$ of £1= £1 12s. $11\frac{8}{11}$d. 3. The shorter course is to add $\frac{7}{8}$ of the sum to thirteen times the sum. The result is £89 6s. $1\frac{17}{64}$d. 4. $164083\frac{1}{3}$. 5. $\frac{1939}{2440}$ of £1. 6. £5 1s. $1\frac{31}{79}$d. 7. $26\frac{23}{50}$s. 8. The quotient $9\frac{153}{175}$ indicates the number of times the latter sum is con-

tained in the former, when both sums are expressed in the same units. 9. The difference is contained $4\frac{323}{423}$ times in the sum. 10. $\frac{35}{432}$ of 1 guinea is equal to 1s. $7\frac{7}{12}$d. 11. $998\frac{1754}{1871}$, an abstract number denoting the number of times the less sum is contained in the greater.

12. Here 17s. 6d. is equivalent to 840 farthings, and 25 new to 24 old farthings, and 1 new is $\frac{24}{25}$ of one old farthing: 840 new $=\frac{24}{25}\times 840=806\frac{2}{5}$ old farthings. $840-806\frac{2}{5}=33\frac{3}{5}$ old farthings loss per week. And $33\frac{3}{5}\times 52=1747\frac{1}{5}$ farthings $=$ £1 16s. $4\frac{4}{5}$d. loss yearly.

XVI.

1. £34 8s. $4\frac{1}{2}$d. 2. £1 gives 1s. 2d. $=\dfrac{7}{6\times 2}$ of £1: £100 gives £$\frac{35}{6}$, or $5\frac{5}{6}$ per cent. 3. First one halfpenny is the gain on three halfpence, the gain is $\frac{1}{3}$ of the capital, and the gain on £100 is £$33\frac{1}{3}$, or $13\frac{1}{3}$ per cent.

Secondly, one halfpenny is the gain on four halfpence, the gain is $\frac{1}{4}$ of the capital, and the gain on £100 is £25, or 25 per cent. The difference is $8\frac{1}{3}$ per cent.

4. £100 bequeathed gives £90 to legatee, or he receives £90 for 100 bequeathed, and £1 for $\frac{100}{90}$, and therefore, £1000 for $\frac{10000}{9}$, or £$1111\frac{1}{9}$ the sum to be bequeathed.

5. First prime cost 3s. 6d. or £$\frac{7}{40}$ gives $4\frac{1}{2}$d. gain: £1 gives $\frac{15}{7}$s. gain, and £100 gives $\dfrac{15\times 100}{7\times 20}$ or £$10\frac{5}{7}$ gain, or $10\frac{5}{7}$ per cent.

Secondly, 3s. $10\frac{1}{2}$d., or £$\frac{37}{160}$, gives $4\frac{1}{2}$d. loss, and £1 gives $\frac{60}{31}$s. loss, and therefore £100 gives $\dfrac{100\times 60}{31\times 20}$ or £$9\frac{21}{31}$ loss, or $9\frac{21}{31}$ per cent.

6. $33\frac{1}{3}$ per cent. 7. $\frac{273}{500}$ of £1. 8. Neither gain nor loss. 9. With £$4\frac{1}{2}$ gain, the prime cost is 100, and £$4\frac{1}{2}$ is contained $\frac{53}{10}$ times in £47 14s. Hence £47 14s. gives $\frac{53}{10}\times 100=$ £530, the prime cost. 10. $21\frac{2}{3}$s. 11. 2s. $11\frac{23}{25}$d. 12. $10\frac{30}{253}$s.

12. Let 1 denote wholesale price to the retailer, who makes a profit of 25 per cent.

$$\text{Then } 1+\frac{25}{100}=1+\frac{1}{4} \text{ or } \frac{5}{4}=16\text{s.}$$

$$\text{therefore } 1=\frac{64}{5}=12\tfrac{4}{5}\text{s.}$$

Next, let 1 denote the manufacturer's price to the wholesale dealer, who makes a profit of 15 per cent,

$$\text{Then } 1+\frac{15}{100}=1+\frac{3}{20} \text{ or } \frac{23}{20}=12\tfrac{4}{5}\text{s.}$$

$$\text{therefore } 1=\frac{256}{23}=11\tfrac{3}{23}\text{s.}$$

Thirdly, let 1 denote the original cost of the article to the manufacturer, who makes a profit of 10 per cent.

$$\text{Then } 1+\frac{10}{100}=1+\frac{1}{10} \text{ or } \frac{11}{10}=11\tfrac{3}{23}\text{s.}$$

$$\text{therefore } 1=\frac{256}{23}\times\frac{10}{11}=\frac{2560}{253}=10\tfrac{30}{253}\text{s.}$$

Or the cost to the manufacturer is $10\frac{30}{253}$s.

XVII.

1. A franc, 10d. $=\frac{5}{6}$s. $=$ £$\frac{1}{24}=\frac{5}{126}$ of 1 guinea. 2. 1 guinea $=504-6=498$ halfpence: and 1 franc $=19$ halfpence. As 498 and 19 are prime to each other. The L.C.M. $=19\times 498$. Hence 19 sovereigns, or 498 francs, are the least numbers of these coins that can be exchanged. 3. $25714\frac{2}{7}$ francs. 4. $8571\frac{3}{7}$ rupees. 5. 10 scudi $=52\frac{1}{2}$ francs, and 1 scudi $=\frac{21}{4}$ of 1 franc; 20 francs $=$ 16s., and 1 franc $=\frac{4}{5}$ of

1s.; $4\frac{1}{6}$s.$=12$ carlini, and 1s.$=\frac{72}{25}$ of 1 carlini. Hence 1 scudi$=\frac{21}{4}$ of $\frac{4}{5}$ of $\frac{72}{25}$ of 1 carlini, and therefore 500 scudi$=500\times\frac{21}{4}\times\frac{4}{5}\times\frac{72}{25}=6048$ carlini. 6. $705\frac{15}{49}$ carlini. 7. $620\frac{40}{193}$ thalers. 8. 100 francs. 9. £6 in London is equivalent to 85 marks in Hamburgh.

XVIII.

1. 17s. $6\frac{4646}{18345}$d. in the pound, and they lose $\frac{11363}{91725}$ of their debt. 2. He loses $\frac{13}{40}$ of £5342 5s. 3. $\frac{5}{8}$ lost, whole debt £120 8s. $4\frac{4}{5}$d. 4. The creditors lose $\frac{287}{480}$ part. The whole debt was £2440. 5. £178 1s. 6d. 6. £810, £621, £729. 7. £$\frac{137\frac{1}{2}}{3}$, £$\frac{78\frac{4}{7}}{2}$, £$\frac{58\frac{13}{14}}{1}$. Each receives $39\frac{2}{7}$ per cent. of his debt. 8. First, £664 was to gain £166, or $\frac{1}{4}$ of the capital employed, which is £25 in the £100, or 25 per cent. Secondly, £830 lost £166, or $\frac{1}{5}$ of the capital employed, or 20 per cent. 9. By dividing £195 into three parts, $\frac{7}{35}$, $\frac{35}{111}$, and $\frac{19}{35}$. 10. £180, £320, £500. 11. Here $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}=\frac{6}{12}+\frac{3}{12}+\frac{2}{12}=\frac{11}{12}$ of the whole, of which $\frac{1}{12}$ share is not assigned. The gain £125 for their share must be deducted from £1,500, which leaves £1,375 to be divided. Their three capitals are denoted by 6, 3, 2, and the first takes 7 per cent. on his 6, the second 5 per cent. on his 3, and the third 3 per cent. on his 2; and the sums of these $42+15+6=63$, or $\frac{2}{3}+\frac{5}{21}+\frac{2}{21}=1$, which gives the shares of the gains £$916\frac{2}{3}$, £$327\frac{8}{21}$, £$130\frac{20}{21}$. 12. £$200\times16\frac{1}{3}$, £$200\times37\frac{1}{6}$, £$200\times53\frac{11}{40}$, £$200\times63\frac{9}{40}$.

XIX.

1. The questions are one the reverse of the other. 2. £333 6s. $2\frac{21}{40}$d. 3. $5\frac{85}{623}$dwts. of pure gold. 4. The supposition gives 96 grains of pure gold, neglecting the alloy. 5. $8\frac{1}{3}$ out of 100 parts, or $8\frac{1}{3}$ per cent. 6. The true weight of a sovereign is 5dwts. $2\frac{1}{4}$grs. The loss on 100 light sovereigns is to be calculated on the supposition of a loss of $1\frac{1}{2}$d. on £3 17s. $10\frac{1}{2}$d., the value of one ounce of standard gold. 7. Neither gain nor loss. 8. 10lbs. of jewellers' gold consists of $5\frac{5}{6}$lbs. of pure gold, and $4\frac{1}{6}$lbs. of alloy, of which alloy $\frac{35}{66}$ of 1lb. added to $5\frac{5}{6}$lbs. pure gold, makes $6\frac{4}{11}$lbs. standard gold; the rest of the alloy, $3\frac{7}{11}$lbs., is neglected. The question now requires the price of $6\frac{4}{11}$lbs. of standard gold at £3 17s. $10\frac{1}{2}$d. per ounce. 9. 17 carats pure gold implies 7 carats alloy, $2\frac{11}{48}$ ounces of pure gold. 10. The sum of £3 17s. $10\frac{1}{2}$d. could have paid £4 13s. 6d. in 1815; but in 1821 the depreciation was 15s. $7\frac{1}{2}$d. on £4 13s. 6d., or $\frac{1}{6}$ of £4 13s. 6d. Hence the depreciation on £100 will be £$16\frac{2}{3}$, or $16\frac{2}{3}$ per cent. 11. If the number of grains in one ounce Avoirdupois be divided by the number of grains in the weight of one sovereign, the quotient will give the number of sovereigns required. 12. The alloy in a sovereign is $10\frac{3}{16}$ grains. The difference will be the excess in value of $10\frac{3}{16}$ grains of silver at 5s. 6d. an ounce, and $10\frac{3}{16}$ grains of copper at 1s. 6d. a pound Avoirdupois of 7,000 grains.

XX.

1. 12dwts. 2. 5s. $3\frac{3}{4}$d. 3. 8dwts. $11\frac{13}{16}$ grains of pure silver, and $13\frac{11}{16}$ grains alloy. 4. $51\frac{611}{624}$ farthings.

5. Supposing the value of the alloy neglected. 80 per cent. of pure silver implies 20 per cent. of alloy, so that $\frac{80}{100}+\frac{20}{100}=1$, or $\frac{4}{5}+\frac{1}{5}=1$. And if the unit be taken as one ounce, one ounce of alloyed silver is in value $\frac{4}{5}$ of an ounce of pure silver, or 4s. And 15 times the value of one ounce of alloyed silver is £3, and £3 is $\frac{3}{4}$ of the value of 1 ounce of pure gold. But three parts of pure gold and one part of copper make up the ounce of alloyed gold. $\frac{1}{4}$ of an ounce of copper, or 25 per cent., must be mixed with 75 per cent. of pure gold.

6. Estimating the coins by the pure silver in them, 1 sestertium$=250\times54$ grains of pure silver, and $\frac{37}{40}$ of 20×24 grains$=37\times12$ grains, which are worth 5s. 2d. The sestertium is worth £7 17s. $1\frac{5}{37}$d.

7. $51\frac{2}{3}$ issued. 8. $12\frac{3}{11}$cwt. of gold, $3\frac{3}{11}$ of silver, and $2\frac{5}{11}$ of copper. 9. Weight of the sovereign $123\frac{171}{623}$ grains, of the half-crown $215\frac{5185}{16921}$ grains. 10. 80 shillings and 5 grains over. 11. $88\frac{287}{356}$ grains.

XXI.

2. $\frac{35}{64}$ of 18cwt. or $\frac{9}{10}$ of one ton. 3. 1520 tons of ore. 4. $20\frac{4}{7}$lbs. and $171\frac{3}{7}$lbs. 5. £860 7s. 2d. 7. $71\frac{1411}{6475}$ pence. 8. $11+100=111 \therefore \frac{11}{111}+\frac{100}{111}=1$. The unit is 24cwt., of which $2\frac{14}{31}$cwt. is tin, and $21\frac{23}{37}$cwt. copper. 9. 75 per cent. of copper and 25 per cent. of tin. 10. $55\frac{104}{125}$lbs. of oxygen, $48\frac{571}{1250}$lbs. of carbon, and $7\frac{889}{1250}$lbs. of hydrogen. 11. $7\frac{159}{343}$ ounces of oxygen, $3\frac{259}{343}$ ounces of sulphur, and $4\frac{268}{343}$ ounces of calcium. 12. 15cwt. of nitre, 2cwt. of sulphur, and 3cwt. of charcoal.

XXII.

1. $44\frac{13}{20}$ lineal inches, $44\frac{13}{20}$ square inches. 2. $44\frac{13}{20}$ cubic inches. 3. $1\frac{69}{235}$ lineal inches; $1\frac{69}{235}$ an abstract number. 4. $1\frac{69}{235}$ square inches; $1\frac{69}{235}$ lineal inches; $1\frac{69}{265}$ an abstract number. 5. The former is double of the latter. 6. $2\frac{1}{2}$ lineal feet. 7. $\frac{28}{165}$ lineal foot. 8. The first quotient is 192 cubic inches, the second 192 square inches, the third 192 lineal inches, and the fourth indicates that 9 cubic inches is contained 192 times in a cubic foot. 9. The difference is $884\frac{13}{16}$ square yards. 10. $306\frac{1}{4}$ square feet. The quotient 9 gives the number of times the latter length is contained in the former. 11. $21\frac{1}{2}$ inches. 12. $452\frac{5}{6}$ square feet. 13. Meadow and arable is $\frac{1}{5}+\frac{3}{8}=\frac{23}{40}$, the rest $\frac{17}{40}=1$ acre, 3 roods, 26 poles$=306$ poles, and $\frac{1}{40}=18$ poles. $\therefore \frac{8}{40}$ or $\frac{1}{5}=144$ poles, meadow: and $\frac{15}{40}$ or $\frac{3}{8}=270$ poles, arable.

14. Here $500+350+800+160=1810 \therefore \frac{50}{181}+\frac{35}{181}+\frac{80}{181}+\frac{16}{181}=1$. In this case the unit is 200 acres, $\therefore \frac{50}{181}$ of 200 acres$=55\frac{45}{181}$, the allotment to the possessor of the rental of £500 a year. In a similar way the rest of the allotments may be found.

XXIII.

1. The length of the measuring rod is $28\frac{7}{82}$ inches, and is contained $98\frac{230}{329}$ times in 77 yards, which is not so near 99 times as by $\frac{99}{329}$ in defect. The distance, therefore, which approaches nearest to 77 yards is 99 times the length of the measuring rod. 2. $1\frac{19}{21}$ inch. 3. $\frac{7}{10}$ of an inch.

4. 5 plethra$=124$ toises, 1 plethra$=\frac{124}{5}$ toises.
9 toises $=$ 17 metres, 1 toise $=\frac{17}{9}$ metres.
35 metres $=$ 39 yards, 1 metre $=\frac{39}{35}$ of 1 yard.
Hence 1 plethra$=\frac{124}{5}$ of $\frac{17}{9}$ of $\frac{39}{35}$ of 1 yard.
1575 plethra$=\frac{1575}{1}\times\frac{124}{5}\times\frac{17}{9}\times\frac{39}{35}=82212$ yards.

5. $3+4+5=12$, $\frac{1}{4}+\frac{1}{3}+\frac{5}{12}=1$. The unit is 492 yards. Hence $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$ of 492 yards are respectively 123, 164, and 205 yards. 6. If the error be *in defect*, the apparent length is 502 yards, and $24\frac{1}{2}$ inches over. If the error be *in excess*, the apparent length is 499 yards and $3\frac{5}{8}$ inches over. 7. $5\frac{1}{2}$ miles is $\frac{11}{15852}$ of the earth's diameter. The height of Mont Blanc above the surface of a globe of 16 inches diameter is represented by $\frac{785}{130778}$ of an inch, or between $\frac{1}{166}$ and $\frac{1}{167}$ of an inch. The height of Kunchingunga above the surface of a globe of 30 inches diameter, is represented by $\frac{14089}{697488}$ of an inch, or between $\frac{1}{49}$ and $\frac{1}{50}$ of an inch.

XXIV.

1. $94\frac{5}{24}$ square yards, $847\frac{7}{8}$ square feet, 10174 square inches. 2. 429 yards. 3. $9\frac{1}{6}$ yards. 4. $10\frac{40}{633}$ feet wide. 5. 280 plots each $1\frac{13}{15}$ foot wide. 6. $266\frac{1}{2}$ yards. 7. $1772\frac{7}{9}$ square yards. 8. $28\frac{1}{2}$ planks. 9. $199\frac{107}{321}$ yards. 10. 25 yards. 11. 1036500 stones. 12. $136\frac{8}{9}$ yards of paper.

XXV.

1. £30 2s. $9\frac{1}{2}$d. 2. 14s. $10\frac{592}{611}$d. per yard. 3. 10 feet high and 16 feet broad. 4. 16 feet broad. 5. £15 14s. $3\frac{31}{48}$d. 6. $6\frac{63}{299}$ feet high. 7. £5 17s. $0\frac{14}{15}$d.

XXVI.

1. $366\frac{1}{6}$ cubic feet. 2. 1728 cubes. 3. 320 cubes. 4. 6 feet long. 5 16 yards. 6. 633600 cubic yards. 7. $334\frac{119}{288}$ cubic feet. 8. $2\frac{1}{12}$ feet long. 9. 960 feet long. 10. $3\frac{2}{3}$ feet deep. 11. 7,800 bricks. 12. The content of the chest is $56\frac{19}{36}$ cubic feet. Quantity of wood used $8\frac{1}{8}$ cubic feet. Difference between the external and internal surfaces of the chest $8\frac{1}{3}$ square feet.

XXVII.

1. 1980lbs. 2. 8190lbs. 3. The weight of a cubic foot of the stone is 160lbs. The weight of the block 235,950lbs. 4. Supposing the internal dimensions given, the external dimensions will be 10ft. 11in., 8ft. 7in., and 5ft. $1\frac{1}{2}$in. The vessel consists of $35\frac{893}{1152}$ cubic feet of iron, of which 4 cubic inches weigh one pound Avoirdupois. 5. The internal surface of the cistern is $143\frac{49}{72}$ square feet, and $17\frac{553}{576}$ cubic feet of lead are required to line it. 6. 1 yard of wire requires $\frac{5}{22}$ of 1lb. of copper, and 50 miles require 4,000lbs. The area of a section of the wire can be found by dividing the number of cubic inches of copper in the wire by its length. 7. 384 feet. 8. $7617\frac{31}{57}$ square yards. 9. 275625 leaves.

XXVIII.

1. Since six men earn £7 6s. 3d. in $7\frac{1}{2}$ days, they earn 19s. 6d. in one day, and one man earns 3s. 6d. in one day. The 10 men will earn £1 15s. in one day, and in $11\frac{3}{4}$ days they will earn £20 11s. 3d. 2. Take an hour as the unit of time.

3. Here 3 men and 5 boys in 4 weeks earn £11:
therefore 3 men and 5 boys in 1 week earn $£2\frac{3}{4}$,
and 6 men and 10 boys in 1 week earn $£5\frac{1}{2}$.
But 7 men and 10 boys in 1 week earn £6:
therefore 1 man in 1 week earns $£\frac{1}{2}$,
and 3 men in 1 week earn $£1\frac{1}{2}$.
Whence 5 boys in 1 week earn $£1\frac{1}{4}$,
and 1 boy in 1 week earns $£\frac{1}{4}$.
Wherefore 5 men and 4 boys earn in 1 week $£3\frac{1}{2}$.
But 5 men and 4 boys earn in some number of weeks, £14.
The time required is $14 \div 3\frac{1}{2} = 4$ weeks.

4. 18 days. 5. Let the work of one child be taken as the unit. Then the work of 9 men, 11 women, and 9 children is equivalent to the work of 71 children. The men receive £1 14s. $4\frac{1}{2}$d., the women £1 5s. $2\frac{1}{2}$d., and the children 5s. 6d.

6. Let the wages of 1 child be taken as the unit.
Wages of 1 man and one child are equivalent to the wages of 4 women.
Wages of 1 man and 3 children are equivalent to the wages of 5 women.
Hence the wages of 2 children are equivalent to the wages of 1 woman,
and the wages of 1 man is equivalent to the wages of $3\frac{1}{2}$ women or 7 children.
Therefore the wages of 1 man, 1 woman, and 3 children, are equivalent to the sum of the wages of 7 children, 2 children, and 3 children, or 12 children.
But the amount of these wages of 12 children is 30s.
The man's wages are 17s. 6d., the woman's 5s., the 3 children 2s. 6d. each.

7. £350 for 7 weeks' work gives £50 weekly. 2 men or three women receive as much as 5 children, therefore 1 man receives as much as $\frac{5}{2}$ children, and 1 woman as $\frac{5}{3}$ children. And 20 men receive as much as 50 children, 40 women as $66\frac{2}{3}$ children, and consequently 20 men, 40 women, and 50 children receive together as much as $166\frac{2}{3}$ children in 1 week. Each child receives $£50 \div 166\frac{2}{3} = 6$s. weekly. 1 woman $\frac{5}{3} \times 6 = 10$s., and 1 man $\frac{5}{2} \times 6 = 15$s.

XXIX.

1. Here 1 man performs $\frac{1}{3000}$ of the work in 1 day, and a certain number of men perform $\frac{2}{5}$ in one day. Hence $\frac{2}{5} \div \frac{1}{3000} = 1200$, the number of men.

2. Since the work of 5 men is equivalent to that of 7 women, the work of 1 woman is denoted by $\frac{5}{7}$ that of a man; and the work of 7 men and 5 women will be denoted by $\frac{75}{7}$ work of a man. The number of days is $34\frac{8}{15}$.

3. The work of two men is equivalent to that of 3 boys, and the work of 9 boys will be equal to that of 6 men, and therefore the work of 12 men and 9 boys will be the same as that of 18 men. 1 man can perform $\frac{1}{450}$ of the work in 1 day, and the work of a boy as $\frac{2}{3}$ of that of a man, or $\frac{8}{675}$ in 1 day. Hence the work of 1 man and 1 boy in 1 day is denoted by $\frac{1}{270}$. The number of days is 30.

4. $30\frac{30}{59}$ days. 5. 28 days. 6. A man in 11 days, a woman in 33 days, and a boy in $38\frac{1}{10}$ days. 7. A man in $31\frac{1}{2}$ days, and a boy in 252 days. 8. 360 days. 9. 4 days.

XXX.

1. If the work be denoted by 1. Then A and B do 1 in 20 days, or $\frac{1}{20}$ in 1 day. B does 1 in 50 days, or $\frac{1}{50}$ in 1 day. Hence A does $\frac{1}{20}-\frac{1}{50}=\frac{3}{100}$ in one day, and $1\div\frac{3}{100}=\frac{100}{3}=33\frac{1}{3}$ days in which A could finish the work by himself.

$$\text{And } A \text{ does } \frac{1}{50} \text{ in 1 day, or in 20 days he does } \frac{20}{50} \text{ or } \frac{2}{5} \text{ of the work.}$$

$$B \text{ does} \frac{3}{100} \qquad ,, \qquad\qquad ,, \qquad\qquad \frac{3\times 20}{100} \text{ or } \frac{3}{5} \quad ,,$$

2. A does 1 in 11 days, therefore $\frac{1}{11}$ in 1 day. B does 1 in 17 days, therefore $\frac{1}{17}$ in 1 day.

Then $\frac{1}{4}+\frac{1}{17}=\frac{28}{187}$. And $1\div\frac{28}{187}=\frac{187}{28}=6\frac{19}{28}$ days.

Together they complete the work in $6\frac{19}{28}$ days.

3. A and B do 1 in 6 days, and $\frac{1}{6}$ in 1 day. B does $\frac{1}{5}$ in $1\frac{1}{2}$ days, and $\frac{2}{15}$ in 1 day. And A does $\frac{1}{6}-\frac{2}{15}=\frac{1}{30}$: and $1\div\frac{1}{30}=30$ days, $1\div\frac{2}{15}=\frac{15}{2}=7\frac{1}{2}$.

That is, A does the work in 30 days, and B in $7\frac{1}{2}$ days.

4. A and B do 1 in 12 days, and $\frac{1}{12}$ in 1 day; similarly A and C do $\frac{1}{15}$ in 1 day, and B and C $\frac{1}{20}$ in 1 day.

$$\text{Then } \frac{1}{12}+\frac{1}{15}+\frac{1}{20} \text{ or } \frac{1}{5} \text{ is double of what } A, B, C \text{ do in 1 day,}$$

$$\text{and therefore } A, B, C \text{ do } \frac{1}{10} \text{ of the work in 1 day, } 1\div\frac{1}{10}=10 \text{ days,}$$

or, together, they finish the work in 10 days.

$$\text{Again } \frac{1}{10}-\frac{1}{12}=\frac{6-5}{60}=\frac{1}{60}, \text{ what } C \text{ does in 1 day.}$$

$$\frac{1}{10}-\frac{1}{15}=\frac{3-2}{30}=\frac{1}{30}, \text{ what } B \text{ does in 1 day.}$$

$$\frac{1}{10}-\frac{1}{20}=\frac{2-1}{20}=\frac{1}{20}, \text{ what } A \text{ does in 1 day.}$$

And A, B, C, separately, do the work in 20, 30, and 60 days respectively.

5. A does 1 in 15 days, and $\frac{1}{15}$ in 1 day. B does 1 in 18 days and $\frac{1}{18}$ in 1 day.

$$\therefore\ A \text{ and } B \text{ together in 3 days do } \frac{1}{5}+\frac{1}{6} \text{ or } \frac{11}{30} \text{ of the work.}$$

$$\text{and } 1-\frac{11}{30}=\frac{19}{30} \text{ of work remains to be done.}$$

Here B leaves, A continues, and is joined by C, who together finish the part left $\frac{19}{30}$ in 4 days. Of this A does $\frac{4}{15}$ in 4 days.

$$\text{And } \frac{19}{30}-\frac{4}{15}=\frac{11}{30} \text{ work left.}$$

$$C \text{ does } \frac{11}{30} \text{ in 4 days, or } \frac{11}{120} \text{ in 1 day, } 1\div\frac{11}{120}=\frac{120}{11}=10\frac{10}{11} \text{ days.}$$

That is, C could do the work by himself in $10\frac{10}{11}$ days.

6. $10\frac{10}{37}$ days. 7. $1\frac{1}{2}$ hrs. 8. $4\frac{14}{19}$ days, $4\frac{10}{20}$ days, $5\frac{5}{21}$ days. 9. 2 days. 10. 23 hrs.

11. 2 days' work of $A=3$ days' work of C;
and 5 days' work of $B=4$ days' work of C.
$\therefore$ 8 days' work of $A=12$ days' work of C;
and 15 days' work of $B=12$ days' work of C.
Hence 8 days' work of $A=15$ days' work of B,
and 1 day's work of $A=\frac{15}{8}$ days' work of B.

Therefore 36 days' work of $A=\dfrac{15\times 36}{8}$, or $67\frac{1}{2}$ days' work of B,

or B will require $67\frac{1}{2}$ days to complete what A can perform in 6 weeks.

12. $3\frac{5}{6}$ days. 13. $3\frac{58}{109}$ days. 14. 5 days.

XXXI.

1. 1728 cubic inches $=1000$ ounces Avoirdupois. 1oz. water $\frac{1}{16}$lb. $=\frac{7000}{16}$ grains. Hence 1 cubic inch of water $=\frac{1000}{1728}\times\frac{7000}{16}=252\frac{109}{144}$ grains Troy. 2. $4\frac{7}{8}$ cubic yards.

3. Content of the cistern $=\dfrac{56}{3}\times\dfrac{31}{3}\times\dfrac{27}{4}=14\times 31\times 3$ cubic feet.

$$1 \text{ cubic foot of water}=1000\text{oz.}=\frac{1000}{16\times 112\times 20} \text{ of 1 ton.}$$

$$\text{Therefore weight of water}=\frac{14\times 31\times 3\times 1000}{16\times 112\times 20}=\frac{31\times 3\times 25}{8\times 8}=\frac{2325}{64}=36\tfrac{21}{64} \text{ tons,}$$

the pressure on the bottom of the cistern, which is at the rate $5\frac{209}{675}$ tons on the square foot.

4. 3402 hogsheads. 5. 120 gallons.

6. Glass A contains 3 parts water + 1 part spirits $=4$ parts.
Glass B contains 4 parts water + 3 parts spirit $=7$ parts.
$\frac{3}{4}$ of water + $\frac{1}{4}$ of spirit $=1$,
and $\frac{4}{7}$ of water + $\frac{3}{7}$ of spirit $=1$,
therefore $1\frac{9}{28}$ of water + $\frac{19}{28}$ of spirit $=2$.
Or the mixture consists of $1\frac{9}{28}$ of water, and $\frac{19}{28}$ of spirit.

7. The first vessel contains $1\frac{399}{625}$ gallons, the second $2\frac{226}{625}$ gallons. 8. $7945\frac{85}{96}$ ounces. 9. 2115lbs. on the square foot, and nine times this pressure on the square yard. 10. 3266lbs. 11. $17\frac{7}{16}$ feet.

XXXII.

1. $\frac{7}{8}$ hour. 2. $8\frac{4}{7}$ minutes. 3. 60 minutes. 4. $\frac{7}{8}$ full.

5. The capacity of the cistern may be represented by 1. Pipe A fills the cistern in 3 hours, and pours in $\frac{1}{3}$ in 1 hour. Pipe B fills it in 4 hours, and pours in $\frac{1}{4}$ in 1 hour. The pipes A and B pour in $\frac{1}{3}+\frac{1}{4}$ or $\frac{7}{12}$ in 1 hour; but pipe C empties the cistern or pours out 1 in 1 hour. Hence the quantity *poured out* being greater than the quantity *poured in* during the same time, the cistern will become *empty* in a certain time. At 3 o'clock, when pipe C is opened, the cistern contains $\frac{2}{3}+\frac{1}{4}$, or $\frac{11}{12}$. And in 1 hour, $1-\frac{7}{12}=\frac{5}{12}$ is excess of quantity *poured out* above that *poured in*. Hence $\frac{11}{12}\div\frac{5}{12}=\frac{11}{5}=2\frac{1}{5}$ hours. The vessel will be empty $2\frac{1}{5}$ hours after 3 o'clock, or at 12 minutes past 5 o'clock.

6. $\frac{1}{12}$ of a day.

7. I am a brazen lion, with fountains in my two eyes, my mouth, and in my right foot. The water flowing from my right eye fills a reservoir in two days, and from the left in three, and from the foot in four. But when the water flows from my mouth, it fills the reservoir in six hours. In what time will the reservoir be filled by the water from the mouth, the eyes, and the foot, if all be opened together?

In $3\frac{33}{85}$ hours the reservoir will be filled.

XXXIII.

1. The difference is $\dfrac{1760\times 3\times 12\times 60}{1769\times 1829}$ minutes, or more than $1\frac{1}{6}$, but less than $1\frac{1}{5}$ of a minute.

2. Here $6+4=10$, and $\frac{3}{5}+\frac{2}{5}=1$, and the unit is 100 miles. Hence *A* walks $\frac{3}{5}$ of 100, or 60 miles, and *B* $\frac{2}{5}$ of 100, or 40 miles, when they meet.

Again $\frac{3}{5}$ of $50=36$ miles, and $\frac{2}{5}$ of $50=20$ miles. When *A* had travelled 30, and *B* 20 miles, they were 50 miles apart. They will a second time be 50 miles apart after having passed each other, when *A* is 30 and *B* 20 miles distant from the places from which *B* and *A* respectively started.

3. 3 hrs. $23\frac{7}{9}$ min. 4. $\frac{1}{10}$ of 1 minute, and $\frac{89}{385}$ of 1 minute. 5. $\frac{75}{88}$ of 1 minute. 6. $28\frac{4}{113}$ seconds through air, $6\frac{174}{235}$ seconds through water, and $4\frac{92}{175}$ seconds through land. 7. 1344 yards.

8. The shooter notes the times of the flight of ball over 500 yards, and of sound over the same distance in 4 seconds. The spectator observes the difference of these to be $2\frac{1}{2}$ seconds. Hence it appears sound moves over 500 yards in $\frac{3}{4}$ second, which gives 1125 the velocity of sound in 1 second.

9. The time required will be the same part of one second as 24900 miles is of 288000; that is, the time is $\frac{24906}{288000}$, or $\frac{83}{960}$ of 1 second, which is between $\frac{1}{11}$ and $\frac{1}{12}$ of a second.

10. $192697\frac{379}{493}$ miles. 11. 24 revolutions. 12. In $11\frac{1}{4}$ hours. 13. 3960 revolutions.

XXXIV.

1. 3hrs. $54\frac{2}{5}$min. p.m.

2. The hour and minute hands of a clock are coincident or in conjunction at 12 o'clock. They are also in conjunction at some point of time in each successive hour (except from 11 to 12 o'clock) until they are again in conjunction at 12 o'clock. The motions of the hour and minute hands are uniform, and the minute hand moves 12 times as fast as the hour hand. Now, after the conjunction at 12 o'clock, the minute hand reaches XII. again when the hour hand points to I. And when the minute hand reaches I. the hour hand will have moved over $\frac{1}{12}$ of the distance between I. and II., and when the minute hand actually overtakes the hour hand, the hour hand will have gone over $\frac{1}{11}$ of the distance between I. and II.; and in the same time the minute hand will have moved over $\frac{12}{11}$. And therefore $\frac{12}{11}$, or $1\frac{1}{11}$, is the space moved over by the minute hand from XII. while the hour hand has moved $\frac{1}{11}$ of the equal space I. to II. And the exact time of the first coincidence of the hands after 12 o'clock, is $1\frac{1}{11}$ hours or $5\frac{5}{11}$ minutes past 1 o'clock. In the same way it may be shewn that the next coincidence will take place at $2\frac{2}{11}$ hours, or at $10\frac{10}{11}$ minutes after 2 o'clock. The successive times of conjunction will be found by adding $1\frac{1}{11}$ hour to the time of each preceding conjunction, reckoning from 12 o'clock 0hr. They are at $1\frac{1}{11}$, $2\frac{2}{11}$, $3\frac{3}{11}$, $4\frac{4}{11}$, $5\frac{5}{11}$, $6\frac{6}{11}$, $7\frac{7}{11}$, $8\frac{8}{11}$, $9\frac{9}{11}$, $10\frac{10}{11}$, $11\frac{11}{11}$ hours respectively.

The hour and minute hands will be exactly opposite, or in opposition, at 6 o'clock, and also once in every hour until they are again in opposition at 6 o'clock.

The hour and minute hands will be at right angles to each other when the minute hand points to XII., and when the hour hand points to III. and to IX., or at 3 and at 9 o'clock; and in every hour there will be two points of time when the two hands are at right angles to one another.

The preceding remarks on the solution of the first question will suggest the mode of proceeding for the exact solution of the second and third questions.

3. See the preceding solution. 4. 4min. 58sec. too slow. 5. At 10hrs. 15min. a.m. on Saturday, the watch is 5min. $36\frac{7}{48}$sec. too slow. 6. The angle is $109\frac{1}{2}$ degrees. 7. The angles at which the hour and minute hand are inclined to the vertical is 60°, when both are equally inclined between the hours of 2 and 3 o'clock. 8. $10\frac{1}{29}$ minutes too slow, and $10\frac{1}{31}$ minutes too fast. 9. 8hrs. 15 min. a.m.

10. There are 11 intervals between 1 and 12 strikes. The interval of two strikes of the first clock is $\frac{35}{11}$ sec., and of the second $\frac{25}{11}$sec., and the seventh strikes take place on the completion of the sixth interval. The times are $\frac{210}{11}$ and $\frac{150}{11}$ seconds, their difference is $\frac{60}{11}$ of 3 seconds, or $\frac{1}{11}$ of 1 minute.

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DECIMALS.

Art. 1. As the local value of figures in the denary scale of notation *increases* tenfold reckoned *from the right towards the left*, the first place being the place of units, the second the place of tens, the third the place of hundreds, the fourth the place of thousands, and so on: it follows that each successive figure *decreases* tenfold when reckoned *from the left towards the right.*

If the scale be continued towards the right and reckoned from the first place of units, the second, third, fourth, &c., places will be the places of tenths, hundredths, thousandths, &c. And if a point be placed before the figure in the place of tenths, this mark will distinguish the fractional from the integral portion of the scale; and while the integral places successively *increase* by tens, hundreds, thousands, &c., the fractional places successively *decrease* by tenths, hundredths, thousandths, &c., from the unit's place.

This extension of the numerical scale reckoning tenfold decrease towards the right constitutes the perfection of the denary system of notation, and renders it complete for expressing the smallest possible fraction as well as the largest possible number.

And further, as these fractions consist only of an extension of the scale below the place of units to tenths, hundredths, thousandths, and so on, the operations of addition, subtraction, multiplication, and division can be performed with these fractions in the same manner as integers, taking care to mark the values of the results by correctly placing the point which separates the fractions from the integers.

2. Def. A decimal fraction or decimal, may be defined to be a fraction the denominator of which is 10, 100, 1000, &c.

There is a peculiar notation asssumed for expressing these fractions. The denominators are omitted, and the numerators only are written with a point placed before that figure, which stands as many places from the right-hand figure as there are ciphers in the denominator. If the number of figures in the numerator be less than the number of ciphers in the denominator, the required number of figures must be made up by prefixing ciphers to the significant figures of the numerator, in order that each figure of the numerator may occupy its proper place in the scale.[1]

---

[1] In order to avoid ambiguity, the point should be placed before the upper part of the first figure of the decimal, and not before the lower part, as a point placed between numbers or symbols in the lower part has been assumed, instead of the symbol ×, to denote that the numbers or symbols are multiplied together.

Thus, the decimal fraction $\frac{5}{10}$ is represented by ·5, $\frac{5}{100}$ by ·05, $\frac{5}{1000}$ by ·005, $\frac{876}{1000}$ by ·876, $\frac{5876}{1000}$ by 5·876.[1]

3. PROP. *The value of any decimal is not altered in value by annexing one, two, three, &c., ciphers to the right hand of it.*

For by annexing 1, 2, 3, &c., ciphers to the right hand of a decimal, both the numerator and the suppressed denominator of the decimal fraction are multiplied by 10, 100, 1000, &c.: and if the numerator and denominator of a fraction be both multiplied by the same number, the fraction is both multiplied and divided by that number, and therefore remains unaltered in value.

Also, if one, two, three, &c., ciphers be found at the right hand of a decimal, they may be omitted without altering the value of the decimal.

For by omitting 1, 2, 3, &c., ciphers, both the numerator and the suppressed denominator of the decimal fraction are divided by 10, 100, 1000, &c.; and when both the numerator and denominator of a fraction are divided by the same number, its value is unaltered.

4. PROP. *To find the sum or difference of two decimals.*

If the decimals be reduced to a common denominator, the sum or difference may be found as the sum or difference of two ordinary fractions.

The sum or difference, however, may be found without reducing the decimal fractions to a common denominator, by arranging the numbers under each other: units under units, tenths under tenths, and so on, and then adding or subtracting as in integers, taking care to place the decimal point in the sum or difference before the place of tenths.[2]

---

[1] If the equivalent notations $\frac{5}{10}$ and ·5 might be named, the former "a decimal fraction," and the latter "a decimal," an ambiguity would sometimes be avoided in speaking of the two forms of the same thing.

A decimal may consist wholly of significant figures, as ·875 is a decimal of three places, consisting of three significant figures; or partly of significant figures, and partly of figures which are not significant, as ·005 is a decimal of three places, consisting of one significant figure.

Any integer may be expressed in the *form* of a decimal fraction, as 25 may be put into the forms $\frac{250}{10}$, $\frac{2500}{100}$, $\frac{25000}{1000}$, &c.

Any decimal may be exhibited as the sum of as many decimals or decimal fractions as there are significant figures in the given decimal.

The decimal ·875 may be exhibited as the sum of three decimal fractions.

$$\begin{aligned} \text{Thus: } \cdot875 &= \frac{875}{1000} = \frac{800 + 70 + 5}{1000} \\ &= \frac{800}{1000} + \frac{70}{1000} + \frac{5}{1000} \\ &= \frac{8}{10} + \frac{7}{100} + \frac{5}{1000} \\ &= \cdot8 + \cdot07 + \cdot005 \end{aligned}$$

[2] Example.—To find the sum and difference of 13·1035 and 7·8105689.

$$\begin{array}{r} 13\cdot1035000 \\ 7\cdot8105689 \\ \hline 20\cdot9140689 \end{array} \text{ sum.} \qquad\qquad \begin{array}{r} 13\cdot1035000 \\ 7\cdot8105689 \\ \hline 5\cdot2929311 \end{array} \text{ difference.}$$

5. Prop. *To multiply one decimal by another, and to deduce the general rule for the place of the decimal point in the product.*

To multiply 3·457 by 21·34:—

$$\text{Here } 3{\cdot}457 = \frac{3457}{1000}, \text{ and } 21{\cdot}34 = \frac{2134}{100}$$

$$\begin{aligned}\text{Then } 3{\cdot}457 \times 21{\cdot}34 &= \frac{3457}{1000} \times \frac{2134}{100} \\ &= \frac{7377238}{100000}, \text{ a decimal of five places,} \\ &= 73{\cdot}77238\end{aligned}$$

Hence the product of two decimals is found by multiplying the decimals as integers, and pointing off from the right-hand figure of the product as many decimal places as there are in the multiplicand and the multiplier.

If, however, there are not as many figures in the product as the number of decimal places in the multiplicand and multiplier together, the required number of decimal places must be made up by prefixing to the product as many ciphers as make up the defect, and the product will be wholly a decimal.

6. Prop. *A decimal is multiplied by* 10, 100, 1000, *&c., by removing the decimal point in the given decimal one, two, three, &c., places towards the right.*

For any given decimal expressed as a decimal fraction is multiplied by 10, 100, 1000, &c., by dividing the denominator by 10, 100, 1000, &c., and thus the decimal fraction, when expressed as a decimal, will consist of 1, 2, 3, &c., decimal places less than before.

7. Prop. *To divide one decimal by another, and to deduce the general rules for the place of the decimal point in the quotient.*[1]

Three ciphers have been annexed to the decimal part of the larger number to make the number of decimal places equal in the two numbers.

It is evident that as ten units make 1 ten, ten tens 1 hundred, and so on, so ten tenths make 1 unit, ten hundredths 1 tenth, and so on; the same law obtains both in the integers and the decimal parts. The processes of addition, subtraction, multiplication, and division of decimals are effected in the same manner as in integral numbers.

A decimal is said to be correct approximately for any number of places when any number of figures on the right of the decimal have been omitted; but if the first of the figures omitted be greater than 5, the last figure on the right must be increased by 1. As an example 5·293 is nearer to 5·2929311 than 5·292, for 5·293 exceeds 5·2929311 by ·0000689, and 5·292 is less than 5·2929311 by ·0009311. Hence the error is less in the former case than in the latter; the former is in excess and the latter in defect of the truth.

[1] In the operations of multiplication and division, as the terms dividend, divisor, and quotient in the latter correspond to product, multiplicand, and multiplier in the former; it is possible that the number of decimals given in a dividend may be

Case 1. When the number of decimals in the dividend is greater than the number in the divisor :—

Let 73·77238 be divided by 3·457.

$$\text{Here } 73{\cdot}77238 = \frac{7377238}{100000}, \text{ and } 3{\cdot}457 = \frac{3457}{1000};$$

$$\begin{aligned} \text{Then } 77{\cdot}77238 \div 3{\cdot}457 &= \frac{7377238}{100000} \div \frac{3457}{1000} \\ &= \frac{7377238}{100000} \times \frac{1000}{5457} \\ &= \frac{2134}{100}, \text{ a decimal of two places,} \\ &= 21{\cdot}34. \end{aligned}$$

Hence, when the number of decimals in the dividend is greater than the number in the divisor, the number in the quotient will be equal to the excess of the number in the dividend above that in the divisor.

Case 2. When the number of decimals in the dividend is equal to the number in the divisor—

Let 7377·238 be divided by 3·457.

$$\text{Here } 7377{\cdot}238 = \frac{7377238}{1000}, \text{ and } 3{\cdot}457 = \frac{3457}{1000};$$

$$\begin{aligned} \text{Then } 7377{\cdot}238 \div 3{\cdot}457 &= \frac{7377238}{1000} \div \frac{3457}{1000} \\ &= \frac{7377238}{1000} \times \frac{1000}{3457} \\ &= \frac{7377238}{3457} \\ &= 2134 \text{ an integer.} \end{aligned}$$

Hence, when the number of decimals in the dividend is equal to the number in the divisor, the quotient will be an integer.

Case 3. When the number of decimals in the dividend is less than the number in the divisor.

Let 737723·8 be divided by ·3457.

$$\text{Here } 737723{\cdot}8 = \frac{7377238}{10}, \text{ and } {\cdot}3457 = \frac{3457}{10000};$$

---

greater than, equal to, or less than the number of decimals in a divisor. Hence there will arise three cases in division of decimals to be considered.

Abbreviated methods have been devised both for the multiplication and division of decimals, whereby the number of figures employed in these operations may be somewhat lessened; and these methods were doubtless very useful in operations requiring a large number of figures. But since the invention of logarithms, these abbreviated methods are of no great practical utility, as such operations can be easily effected by the use of logarithmic tables.

$$\begin{aligned}
\text{Then } 737723{\cdot}8 \div {\cdot}3457 &= \frac{7377238}{10} \div \frac{3457}{10000} \\
&= \frac{7377238}{10} \times \frac{10000}{3457} \\
&= \frac{7377238 \times 1000}{3457} \\
&= \frac{7377238000}{3457} \\
&= 2134000 \text{ an integer.}
\end{aligned}$$

Hence, when the number of decimals in the dividend is less than the number in the divisor, a number of ciphers equal to the defect must be annexed to the dividend, and the quotient will be an integer.

If there be a remainder after all the ciphers of the dividend have been employed, the division may be continued by annexing other ciphers, considered as decimals, to the dividend, either until the division terminate, or until any required number of decimals in the quotient is obtained.[1]

8. PROP. *A decimal is divided by 10, 100, 1000, &c., by removing the decimal point one, two, three, &c., places farther towards the left of the given decimal.*

For any given decimal expressed as a fraction is divided by 10 100, 1000, &c., by multiplying the denominator of the decimal fraction by 10, 100, 1000, &c., and thus the decimal fraction, when expressed as a decimal, will consist of 1, 2, 3, &c., decimal places more than before.

9. PROP. *To reduce an ordinary fraction to a decimal.*

If the numerator and denominator of any ordinary fraction be both multiplied by 10, 100, 1000, &c., and the numerator so increased be divided by the denominator of the given fraction, the result will be the equivalent decimal fraction.

In practice, this may be more readily effected by annexing one, two, three, &c., ciphers, as decimals to the numerator, and dividing by the denominator of the fraction, the quotient will be the equivalent decimal, consisting of as many places as there were ciphers annexed to the numerator.

In the conversion of ordinary fractions into their equivalent decimals, it will be found that in some cases the process of division will terminate, and in others it will not terminate. Each class of decimals will require separate considerations in the converse process.[2]

---

[1] It will be found sometimes convenient in mathematical operations to change a multiplier into a divisor, and conversely. For instance, since $\frac{1}{2} = {\cdot}5$, it follows that to multiply by ·5 is the same as to multiply by $\frac{1}{2}$ or divide by 2, and conversely. And in the same way, to divide by ·625 is the same as to multiply by $\frac{8}{5}$.

[2] For examples, in the conversion of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$ to their equivalent decimals, as the denominators respectively consist of one, two, three factors; one, two, three ciphers

10. **Prop.** *To determine what must be the prime factors of the denominators of fractions which can be converted into terminating decimals.*

---

respectively must be annexed to the respective numerators, and the equivalent decimals will terminate and consist of one, two, three places respectively.

$$\text{Thus } \frac{1}{2}=\frac{1\times10}{2\times10}=\frac{10}{2\times10}=\frac{5}{10}=\cdot5.$$

$$\frac{3}{4}=\frac{3\times100}{4\times100}=\frac{300}{4\times100}=\frac{75}{100}=\cdot75.$$

$$\frac{5}{8}=\frac{5\times1000}{8\times1000}=\frac{5000}{8\times1000}=\frac{625}{1000}=\cdot625.$$

In the first 100 natural numbers there are only 13 which are composed of factors of 2 and of 5 ; it is obvious, therefore, that of the series of fractions whose denominators are the first 100 natural numbers (omitting unity) there will be 13 which produce terminating decimals, and 86 repeating decimals. Hence it is clear that the greater number of fractions when reduced will produce repeating decimals. In practice, however, repeating decimals are not generally employed, except in cases where very minute accuracy is required in the result. In ordinary cases any approximate degree of accuracy may be attained by considering repeating decimals to terminate at any figure in the series of repeating figures, and the limit of error in excess or defect can be definitely stated in each case. And the larger the number of decimal figures be taken, the more nearly will the terminating decimal approximate to the exact value of the repeating decimal.

If $\frac{1}{7}$ be converted into its equivalent decimal,

$$\frac{1}{7}=\frac{1\times1000000}{7\times1000000}=\frac{1000000}{7\times1000000}=\frac{142857\frac{1}{7}}{1000000}=\frac{142857}{1000000}+\frac{1}{1000000}\cdot\frac{1}{7}$$

it appears that 6 figures continually recur, and if 1, 2, 3, 4, 5, &c., figures be taken successively as approximate values, it will appear how the successive terminating decimals differ from the true value of the repeating decimal.

Here $\frac{1}{7}$ = ·142857142857142857 . . . .

Let ·1, ·14, ·142, ·1428, ·14285, ·142857, &c., be successively taken as approximate values of the recurring decimal.

If ·1 be taken for $\frac{1}{7}$, then $\frac{1}{7}$ is greater than ·1, but less than ·2.

·14	·14	·15.
.142	·142	·143.
·1428	·1428	·1429.
·14285	·14285	·14286.
·142857	·142857	·142858.
·1428571	·1428571	·1428572.

and so on. And the limits of error both in excess and defect from the true value can be determined in every case.

Let the first assumption be taken: $\frac{1}{7}$ greater than ·1, but less than ·2.

$$\text{Then } \frac{1}{7}-\cdot1=\frac{1}{7}-\frac{1}{10}=\frac{3}{70}\text{ error in defect.}$$

$$\cdot2-\frac{1}{7}=\frac{2}{10}-\frac{1}{7}=\frac{4}{70}\text{ error in excess.}$$

Let the third be taken: $\frac{1}{7}$ greater than ·142, but less than ·143.

$$\frac{1}{7}-\cdot142=\frac{1}{7}-\frac{142}{1000}=\frac{6}{7000}\text{ error in defect.}$$

$$\cdot143-\frac{1}{7}=\frac{143}{1000}-\frac{1}{7}=\frac{1}{7000}\text{ error in excess.}$$

If one, two, three, &c., ciphers be annexed to the numerator, it becomes a multiple of 10, 100, 1000, &c., and those numbers are exactly divisible by 2 and by 5, by 2×2 and by 5×5, by $2 \times 2 \times 2$ and by $5 \times 5 \times 5$, &c.

Hence it follows that when the denominator is 2, 4, 8, &c., or 5, 2 125, &c., the fraction can be reduced to a terminating decimal, and the number of ciphers to be annexed to the numerator is equal to the number of factors of 2 or of 5 contained in the denominator.

11. Prop. *In the conversion of an ordinary fraction into a decimal, if the division of the numerator with ciphers annexed, as decimals, by the denominator do not terminate, the remainders will recur in periods, each of not more figures than the units in the denominator, less by one.*

Let the seventh be taken: $\frac{1}{7}$ greater than ·1428571, but less than ·1428572.

$$\frac{1}{7} - \cdot 1428571 = \frac{1}{7} - \frac{1428571}{10000000} = \frac{3}{70000000} \text{ error in defect.}$$

$$\cdot 1428572 - \frac{1}{7} = \frac{1428572}{10000000} - \frac{1}{7} = \frac{4}{70000000} \text{ error in excess.}$$

It is seldom that more than seven decimal places are required in computations; and this number would make the approximation correct to tenths of a million of the unit.

In converting an ordinary fraction into a repeating decimal, in which the number of repeating figures is large, the labour may be much shortened by a method given by Mr. Colson in p. 162 of Newton's Fluxions. He gives as an example the conversion of $\frac{1}{29}$ into it corresponding repeating decimal. If 1 with ciphers annexed be divided by 29 for any number of figures in the quotient, then $\frac{1}{29} = \cdot 03448\frac{8}{29}$ (1); and multiplying these equals by 8, $\frac{8}{29} = \cdot 27584\frac{64}{29} = \cdot 27586\frac{6}{29}$.

If this value of $\frac{8}{29}$ be substituted in the expression (1)

then $\frac{1}{29} = \cdot 0344827586\frac{6}{29}$. (2)

Again, multiplying these equals by 6,

$$\frac{6}{29} = \cdot 2068965516\tfrac{36}{29} = \cdot 2068965517\tfrac{7}{29};$$

substitute this value of $\frac{6}{29}$ in the expression (2);

$$\therefore \frac{1}{29} = \cdot 03448275862068965517\tfrac{7}{29}. \quad (3)$$

Thirdly, multiplying these equals by 7,

then $\frac{7}{29} = \cdot 241379310344827586 19\frac{49}{29} = \cdot 24137931034482758620\frac{20}{29}$;

and substituting this value of $\frac{7}{29}$ in the expression (3).

$$\frac{1}{29} = \cdot 034482,758620,689655,172413,793103,448275,8620\tfrac{20}{29}.$$

It appears from this result, that the repeating period consists of 28 figures.

$$\text{And } \frac{1}{29} = \cdot\dot{0}34482,758620,689655,172413,793\dot{1}$$

is the equivalent repeating decimal required.

In the conversion of ordinary fractions to decimals, when the denominators are large numbers, the labour may be considerably lessened by means of a table of the reciprocals of the natural numbers. In Barlow's Mathematical Tables, the reciprocals of the natural numbers from 1 to 10,000 have been calculated to ten places of decimals. By means of this table, the approximate decimal corresponding to any ordinary fraction may be more readily calculated when the denominator is a large number, than by ordinary division.

To convert $\frac{45}{6779}$ into a decimal. The table gives $\frac{1}{6779} = \cdot 000147544$, and multiplying these equals by 45, then $\frac{45}{6779} = \cdot 0001475144145$.

It is obvious that the multiplication of the reciprocal of 6779 by 45, is a less laborious process than dividing 45 with ciphers annexed by 6779.

As the division does not terminate; the remainder at each step of the division is greater than 0, but less than the denominator. Consequently, there cannot be more different remainders than one less than the denominator, while the number is without limit. Hence, after as many remainders as are equal to the number in the denominator less by unity, a remainder will recur the same as some former one; after this, the circumstances producing the succeeding remainders are the same as those which produced the preceding ones, and therefore the second period of remainders will be the same as the first. And, similarly, the third, fourth, &c., periods will also be identical to the first period. From that step in the division where a remainder becomes the same as a former one, it is evident from the nature of the division that the former figures of the quotient must be repeated in the same order.

Decimals are called repeating, recurring, or circulating decimals, in which one or more figures continually recur, and the figure or period of figures that recur, or are repeated, are called repetends.

The periods or repeating figures in decimals are distinguished by placing a point over the first and last figures of the periods. If one figure only be repeated, one point only is placed over the repeating figure.

If the decimal consist only of figures which recur, it is named a pure repetend; if it consist partly of figures which do not, and which do recur, it is named a mixed repetend:—thus, $\cdot\dot{3}$, $\cdot\dot{4}\dot{5}$ are pure repetends, and $\cdot 1\dot{6}$, $\cdot 34\dot{5}6\dot{7}$ are mixed repetends.

12. Prop. *To convert any given terminating decimal into an ordinary fraction.*

A given decimal can be expressed as a decimal fraction by writing under the numerator the suppressed denominator. And if the numerator and denominator of this decimal fraction be divided by their greatest common measure, the quotients will be the numerator and denominator of the fraction equivalent to the given decimal.

13. Prop. *To change a pure repeating decimal to its equivalent ordinary fraction.*

If 1 with ciphers annexed as decimals be divided by 9, 99, 999, 9999, &c., respectively,

Then $\frac{1}{9} = \cdot 11111111 \ldots\ldots\ldots$ or $\cdot\dot{1}$
$\frac{1}{99} = \cdot 01010101 \ldots\ldots\ldots$ or $\cdot\dot{0}\dot{1}$
$\frac{1}{999} = \cdot 001001001 \ldots\ldots\ldots$ or $\cdot\dot{0}0\dot{1}$
$\frac{1}{9999} = \cdot 00010001 \ldots\ldots\ldots$ or $\cdot\dot{0}00\dot{1}$
&c., &c., &c.

and any other numbers, 2, 3, 4, &c., when divided by 9, 99, 999, 9999, &c., respectively, will give the same number of figures continually repeated. Hence, conversely: Every pure repeating decimal consisting of 1, 2, 3, 4, &c., repeating figures respectively is equal to a fraction having the repeating figures for the numerator and as many nines for the denominator.

A singular exception arises when 9 is the repeating figure of the decimal. In this case $\frac{9}{9}$ is not a fraction, and the repeating decimal cannot have been produced by the division of any number with ciphers annexed, by any other number.

14. Prop. *To reduce a mixed repeating decimal to its equivalent ordinary fraction.*

Every mixed repeating decimal as $\cdot 12\dot{3}$ can be separated into two parts, of which one is, and the other is not, repeating.

$$\text{Here } \cdot 12\dot{3} = \cdot 12 + \cdot 00\dot{3}$$

$$\cdot 12 = \frac{12}{100}, \text{ and } \cdot 00\dot{3} = \frac{1}{100} \text{ of } \cdot\dot{3} = \frac{3}{900}$$

$$\text{and } \cdot 12 + \cdot 003 = \frac{12}{100} + \frac{3}{900} = \frac{108}{900} + \frac{3}{900}$$

$$= \frac{111}{900} = \frac{123-12}{900}$$

$$\text{Therefore } .12\dot{3} = \frac{123-12}{900}$$

Hence, any mixed repeating decimal is equivalent to the fraction whose numerator is the difference between the numbers composed of the mixed repeating decimal and of the non-repeating figures, and whose denominator consists of the number composed of as many nines as the figures in the repeating part prefixed to as many ciphers as figures in the non-repeating part.

15. Prop. *To add and subtract repeating decimals.*

If the repeating periods of the given decimals do not begin with figures of the same local value, each period must be made to begin with a figure of the same local value and to end with one of the same lower local value, so that the decimals may have the same number of repeating figures, and of the same local values. The sum or difference may then be found, and the number of repeating figures in the sum or difference will be the same as those noted in the decimals.[1]

---

[1] Example.—To find the sum and difference of $12\cdot4\dot{5}\dot{3}$ and $7\cdot\dot{2}7\dot{6}$.

Here $12\cdot4\dot{5}\dot{3}$ contains two repeating figures beginning from the place of hundredths, and $7\cdot\dot{2}7\dot{6}$ contains three, beginning from the place of tenths. The repeating figures in $7\cdot\dot{2}7\dot{6}$ must be made to begin from the place of hundredths, making the first figure non-repeating, thus $7\cdot2\dot{7}6\dot{4}$, so that 764 are now made the repeating figures. It will be obvious that as the repeating figures in each number *begin* with a figure of the same local value, and as there are 3 in one and 2 in the other, there must be 6 repeating figures in each, that they may end with figures of the same but lower local value.

$$\begin{array}{llll} \text{Thus, } 12\cdot4\dot{5}\dot{3} \text{ must be written} & 12\cdot4\dot{5}3535\dot{3}5 & & 12\cdot4\dot{5}3535\dot{3}5 \\ \quad 7\cdot\dot{2}7\dot{6} \quad ,, \quad ,, & \underline{7\cdot2\dot{7}6276\dot{2}7} & & \underline{7\cdot2\dot{7}6276\dot{2}7} \\ & 19\cdot7\dot{2}9811\dot{6} \text{ sum.} & & 5\cdot1\dot{7}7259\dot{0} \text{ difference.} \end{array}$$

It may be observed that one or two figures of each repetend must be continued in order to secure the accuracy of the last repeating figure in the result.

Or, the given repeating decimals may be converted into ordinary fractions, and the sum or difference of the fractions found, and then the result may be converted into a decimal if required.

16. Prop. *To multiply and divide repeating decimals.*

To multiply or divide one repeating decimal by another, or a terminating decimal by a repeating decimal, is a tedious process. In general it will be found more convenient to reduce the repeating decimals to ordinary fractions, and find the product or quotient of the fractions, and afterwards, if necessary, to convert the fractional product or quotient to its equivalent decimal.[1]

17. Prop. *To find the value of the decimal of any concrete quantity in smaller units of the same kind: and conversely, to reduce any concrete quantity or decimal to the decimal of any other unit of the same kind.*

Concrete decimals are subject to the same rules of operation as ordinary concrete fractions.

A concrete decimal of a larger unit is reduced to a number or decimal of a smaller unit of the same kind by multiplying the given decimal by the number of smaller units contained in the larger.

And, conversely, a concrete number, or decimal of a smaller unit, is changed into the decimal of a larger unit of the same kind, by dividing by as many of the smaller units as make one of the greater.

If any of the given decimals be repeating decimals, it will be found more convenient to change them into ordinary fractions, and employ them instead of the repeating decimals.[2]

18. Prop. *To find the sum or difference of two concrete decimals of the same kind.*

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[1] Example.—Multiply and divide $2\cdot\dot{2}\dot{7}$ by $4\cdot1\dot{6}$.

$$\text{Here } 2\cdot\dot{2}\dot{7} = 2\frac{27}{99} = 2\frac{3}{11} - \frac{25}{11},$$

$$\text{and } 4\cdot1\dot{6} = 4\frac{15}{90} = 4\frac{1}{6} = \frac{25}{6}.$$

$$\text{Then } 2\cdot\dot{2}\dot{7} \times 4\cdot1\dot{6} = \frac{25}{11} \times \frac{25}{6} = \frac{625}{66} = 9\cdot4\dot{6}\dot{9}, \text{ the product,}$$

$$\text{and } 2\cdot\dot{2}\dot{7} \div 4\cdot1\dot{6} = \frac{25}{11} \times \frac{6}{25} = \frac{6}{11} = \cdot\dot{5}\dot{4}, \text{ the quotient.}$$

It may be remarked that sometimes it will be found that the product or quotient of two repeating decimals may produce a terminating decimal.

[2] Example.—What is the value of £·828125 in units of lower denomination?

The decimal of a pound is reduced to shillings by multiplying by 20, and 6 decimal places marked off in the product gives 16 shillings.

Next, the decimal of a shilling is reduced to pence by multiplying by 12, and 6 decimals marked off gives the pence 6.

And lastly, the decimal of a penny is reduced to farthings by multiplying by 4, and 6 places marked off give the farthings 3.

The process may be thus shown :—

The value of each of the concrete decimals may be found in smaller units of the same kind, and then their sum or difference.

Or if the given decimals be of different units, they may be reduced to equivalent decimals of the same unit, and the sum or difference can be found as in abstract decimals.

If, however, any of the decimals be repeating decimals, these may, with greater convenience, be reduced to their equivalent ordinary fractions before performing the operations.

19. Prop. *To find the product or quotient of two concrete decimals.*

If the given decimals be terminating decimals, the product or quotient can be found in the same way as if the decimals were abstract. If one or both of them be repeating decimals, it will be found necessary in all cases of multiplication or division, first to reduce the given repeating decimals to their equivalent ordinary fractions, and then to perform the operation required.

If one of the given decimals be concrete and the other abstract, or if both be concrete, the nature of the product or quotient will be determined by the same considerations as the product or quotient of two concrete integers or ordinary fractions.

£·828125
20
s. 16·562500
12
d. 6·750000
4
q. 3·000000 so that £·828125 = 16s. 6¾d.

Conversely.—To express 16s. 6¾d. as the decimal of one pound.

First: 3 farthings with ciphers annexed as decimals, divided by 4, gives the decimal ·75 of 1 penny.

Next: 6·75*d*, with ciphers annexed as decimals, divided by 12, gives ·5625 of 1 shilling.

And 16·5625*s*., with ciphers annexed, divided by 20, gives ·828125 of 1 pound.

The process may be exhibited thus:—

4)3·00
12)6·7500*d*.
20)16·562500*s*.
£·828125

In all other concrete quantities which according to their nature are divisible into larger and smaller units, by a similar method may be found the units of a smaller unit in any decimal of a larger unit; and conversely, any smaller units may be reduced to the decimal of any larger unit of the same kind.

Example.—What is the exact value of £·5$\dot{1}\dot{6}$ sterling?

$$\text{Here } £\cdot 5\dot{1}\dot{6} = \cdot 5\dot{1}\dot{6} \text{ of } £1 = \frac{516 - 51}{900} = \frac{465}{900} = \frac{31}{60} \text{ of } £\mathbf{1}.$$

$$\text{And } \frac{31}{60} \text{ of } £1 = \frac{31}{60} \times \frac{20}{1} = \frac{31}{3} \text{ of 1s.} = \text{10s. 4d.}$$

EXERCISES.

Abstract Decimals.

I.

1. Shew that decimal fractions are simply an extension of the common denary system of notation adopted in the arithmetic of whole numbers.

2. Write fully in words the value of each figure in 321·123.

3. Define a decimal fraction, and a decimal; and explain what notation is used to distinguish decimals from integers.

4. What are the advantages of decimal fractions over ordinary fractions? State why decimal fractions are not always used in calculations.

5. Explain whether ·067 or ·068 is more nearly equal to ·06748, and express in words the error in excess or defect in each case.

II.

1. Find the sums of the following finite decimals, and express the results in words:—

(1.) 100·1, ·0001, 125·125 and 500.
(2.) ·172, ·06,1·0004, 20·02 and ·001.
(3.) 12·4873, 124·873, 1248·73, 12487·3, 1·24873 and ·124873.
(4.) 270, ·072, ·0001, 100·3508, 5·7764, and ·12345.
(5.) ·0010375, ·0006875, ·0008125, ·0059625, ·0000718, ·0009282.

2. Find the sum of all the decimals which can be formed with the digits 0, 3, 7, 9 in every possible way, all the four digits being used in the same number.

III.

Find the differences of the following finite decimals:—

·02 and ·002: ·006 and ·007: ·027 and ·123: 20·204 and 2·91741: ·027541 and ·207514: 234·9 and ·99999: ·100 and ·001: 214·0007 and 1000·000007: ·0078125 and ·00048828625.

IV.

1. Shew that any decimal is multiplied by 1000 by removing the decimal point in the multiplicand three places towards the right.

2. Devise a rule for the multiplication of finite decimals depending on the local value of the figures without reference to fractions.

3. Find the products of the following finite decimals:—

2·5 and 3·7: ·025 and 8: 1·25 and 12: 421 and ·002: ·01 and 100000: ·00001 and 10000: 75·86425 and 10000: 3·45 and 34·5: ·000125 and 125000: ·007853 and ·00476: ·0015625 and ·00064:

37·0004 and 2·2365: ·09183475 and 365·64: ·0001125 and 3·25: 56·001 and 1·002: ·00003712 and ·00078: ·02534 and ·03256.

4. Multiply the sum of 2·616, ·00132, and 1·0448 by ·62639.

V.

Find the quotients of the following finite decimals:—

1. ·5 by 5: ·075 by ·25: ·00081 by ·27: ·0175 by 3·5: ·00385 by 35: ·000125 by 125000: ·002291 by ·29: 21·875 by 17·5: ·00426 by ·071: ·000625 by ·125: 552·5325 by 3·25: 43·7265 by 2·37: 278·48352 by 27·84.

2. ·94 by ·47: ·144 by ·012: 2·625 by ·005: ·001813 by ·000049: 14·904 by ·324: 4·8822 by ·0079: 2·1825 by ·0025: 22·801 by ·151.

3. ·72 by ·018: 2·3 by ·000625: 4 by ·00001: 400 by ·08: ·0021 by ·00003: 126·025 by ·0000071: 514·25 by ·00000085: 10 by ·00625.

4. Divide ·0175 by 35, and 35 by ·0175; 1700 by ·0017, and ·0017 by 1700; ·05 by ·02, and ·02 by ·05; ·000282 by ·000705, and ·000705 by ·000282.

5. Prove that a decimal is divided by 10000 by removing the decimal point in the dividend four places towards the left.

6. Enunciate the general rules for the division of decimals. In cases where the division does not terminate, explain how to determine the place of the decimal point in the remainder.

7. Find the quotients of the following decimals, each to seven places of decimals, and state the values of the remainders in each case:—

148 by ·00727: 16·65 by 2·927: 72 by 1·00005: ·0006163 by 2994·418: 1 by 3·14159: 639·176 by 17·23: 9·91931 by 35·3: ·0001 by ·03: 41·22054 by ·00087009.

8. Find the quotient arising from dividing the sum of 30·33 and ·0337 by their difference.

9. Find the product of the quotients of 5·625 divided by ·45, and ·045 divided by 56·25.

10. Divide 375 by ·75 and ·75 by 375, and find the sum and difference of the quotients.

11. Divide 76·57 by ·0019 and multiply the quotient by $\frac{3}{7}$ of ·0008568.

12. Express the millionth part of ·5 as a decimal.

VI.

1. Convert the divisor ·625625 into a multiplier.

2. What divisor is equivalent to ·0064 as a multiplier?

3. By what number must ·0016 be multiplied that the result may be unity?

4. How many times can ·0087 be taken from 2·291? What fraction will the remainder be of the former?

5. Divide ·31398 by ·079, and write down the true value of the remainder after obtaining two decimal places in the quotient.

6. Find the greatest common divisor of 36·595 and 57·980.

7. What is the least number which can be divided by each of the numbers 1·36652 and 246·8642?

VII.

1. In what cases can an ordinary fraction be expressed by a *finite* decimal? Shew that the number of decimal places in such cases may be inferred from the factors of the denominator.

2. Of the fractions which produce terminating decimals, write down the denominators of such fractions as will produce decimals of 1, 2, 3, 4, 5, 6, 7 places respectively.

3. Convert the following fractions into decimals, and express in words at length the values of the respective decimals:—

$\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{5}{16}, \frac{5}{32}, \frac{23}{640}, \frac{17}{125}, \frac{7}{62500}, \frac{3476}{15625}, \frac{98}{28672}, \frac{372}{12500}, \frac{323}{1280}.$

And conversely, reduce the following decimals to ordinary fractions in their lowest terms:—

·5, ·75, ·625, ·3125, ·15625, ·0359375, ·176, ·000112, ·318464, ·00341796875, ·02176, ·25234375.

4. Calculate the limits of the error made in taking $\frac{355}{113}$ as an approximate value of 3·1415926 to seven places of decimals.

VIII.

Calculate the following expressions, each to 7 places of decimals:—

1. $\frac{80}{81}+\frac{81}{82}+\frac{82}{83}+\frac{83}{84}.$

2. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}.$

3. $1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\cdots\cdots\frac{1}{1.2.3.4.5.6.7.8.9}.$

4. $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\frac{1}{3^6}.$

5. $\frac{1}{10^9}\left\{1-\frac{4}{10^2}+\frac{4.5}{1.2}\cdot\frac{1}{10^4}-\frac{4.5.6}{1.2.3}\cdot\frac{1}{10^6}+\frac{4.5.6.7}{1.2.3.4}\cdot\frac{1}{10^8}\right\}.$

6. $\frac{4}{239}\left\{\frac{1}{5}-\frac{1}{3\times5^2}+\frac{1}{5\times5^5}-\frac{1}{7\times5^7}+\frac{1}{9\times5^9}\right\}.$

IX.

1. What is meant by *pure* and by *mixed* circulating decimals? Shew how to ascertain from knowing the factors of the denominators of fractions, whether they produce pure or mixed circulating decimals. Is it indifferent whether the numerators are prime or composite numbers? Confirm the answer by examples.

2. Shew that in the reduction of a fraction to a decimal, if the figures of the decimal do not terminate, they must recur in periods of one, two, three, four, &c., figures, which are always less in number than the denominator.

3. Change the following fractions into their equivalent decimals:—
$\frac{1}{3}$, $\frac{4}{7}$, $\frac{5}{9}$, $\frac{4}{11}$, $\frac{11}{13}$, $\frac{1}{17}$, $\frac{5}{37}$, $\frac{15}{41}$, $\frac{8}{53}$.
And conversely, reduce the following *pure repeating* decimals to their equivalent fractions in the lowest terms:—

$\cdot\dot{3}$, $\cdot\dot{5}7142\dot{8}$, $\cdot\dot{5}$, $\cdot\dot{4}\dot{5}$, $\cdot\dot{8}4615\dot{3}$, $\cdot\dot{0}58823529411764\dot{7}$, $\cdot\dot{1}3\dot{5}$, $\cdot\dot{3}658\dot{5}$, $\dot{1}50943396226\dot{4}$.

4. Reduce the following fractions to decimals:—
$\frac{1}{6}$, $\frac{11}{12}$, $\frac{1}{48}$, $\frac{19}{175}$, $\frac{71}{105}$, $\frac{11}{21}$, $\frac{189}{925}$, $\frac{1}{1998}$, $\frac{5359}{198000}$.
And conversely, convert the following *mixed repeating* decimals into equivalent fractions in the lowest terms:—

$\cdot1\dot{6}$, $\cdot91\dot{6}$, $\cdot0208\dot{3}$, $\cdot10\dot{7}1428\dot{5}$, $\cdot\dot{6}76190\dot{4}$, $\cdot\dot{5}23809\dot{5}$, $\cdot20\dot{4}3\dot{2}$, $\cdot0\dot{0}0\dot{5}$, $\cdot0270\dot{6}\dot{5}$.

5. Shew that $\frac{1}{29}=\cdot03448\frac{8}{29}$; hence, without any further division, find the value of $\frac{27}{29}$ to ten places of decimals.

X.

1. In the addition and subtraction of two or more repeating decimals, how may the number of repeating figures in the sum or difference be ascertained? How many repeating decimals will be in the sum and difference of two decimals, one of which contains four, and the other six repeating figures?

2. Find the sums of the following repeating decimals:—

$12\cdot1\dot{6}$, $110\cdot3\dot{5}\dot{4}$ and $1\cdot\dot{1}41\dot{2}$: $\dot{3}$, $\cdot\dot{1}2\dot{3}$ and $\cdot\dot{5}67\dot{8}$:

$6\cdot2\dot{7}$, $18\cdot\dot{5}6\dot{1}$, $12\cdot3\dot{4}\dot{5}$ and $1\cdot0001$: $1\cdot0\dot{3}$, $2\cdot00\dot{3}$ and $3\cdot000\dot{3}$:

$10\cdot1\dot{6}$, $300\cdot\dot{8}5714\dot{2}$, $3\cdot\dot{0}9523\dot{8}$, $\cdot07\dot{3}22\dot{4}$, and $1\cdot01\dot{1}8564\dot{0}$:

$257\cdot30\dot{6}\dot{1}$, $5\cdot\dot{6}81\dot{9}$, $\cdot\dot{1}86731\dot{2}$ and $\cdot\dot{1}5359\dot{8}$:

$10\cdot12\dot{4}7\dot{3}$, $62\cdot91$, $400\cdot00\dot{0}\dot{3}$ and $\cdot\dot{2}56\dot{4}$:

$1\cdot\dot{5}$, $4\cdot0\dot{2}\dot{3}$, $12\cdot\dot{8}4\dot{5}$ and $\cdot\dot{2}42032\dot{9}$:

$100\cdot001\dot{7}$, $2\cdot38457789$, $109\cdot200471385$ and $3.\dot{4}5\dot{6}$.

3. Shew that nine times the sum of $\cdot\dot{7}\dot{3}$ and $\cdot\dot{3}\dot{7}$ is an integer.

XI.

1. Find the differences of the following repeating decimals, and express the results as fractions in their lowest terms.

$3\cdot\dot{2}\dot{7}$ and $4\cdot1\dot{6}$: $\cdot\dot{6}$ and $\cdot\dot{2}9\dot{6}$: $\cdot\dot{7}3462\dot{5}$ and $\cdot\dot{3}0719\dot{6}$: $\cdot61\dot{2}\dot{3}$ and $\cdot041\dot{6}$:

$\cdot3\dot{6}3\dot{9}$ and $1\cdot02\dot{3}7128\dot{0}$: $\cdot089\dot{2}8571\dot{4}$ and $\cdot0\dot{9}$: $14\cdot\dot{2}57\dot{8}$ and $3\cdot1\dot{4}\dot{5}$.

$\cdot340\dot{2}\dot{7}$ and $\cdot12\dot{7}$: $\cdot1\dot{2}\dot{3}$ and $\cdot1\dot{2}3\dot{4}$: $\cdot019\dot{3}\dot{4}$ and $\cdot27\dot{3}24\dot{2}$.

2. Reduce $\cdot265625$ and $\dot{7}1428\dot{5}$ to fractions, and find their difference.

XII.

1. Find the products of the following decimals:—

83 and $\cdot041\dot{3}$: $\cdot\dot{2}9\dot{7}$ and 12000: $412\cdot\dot{4}2\dot{1}$ and $\cdot002$: $\cdot41716\dot{2}6984\dot{1}$ and $\cdot0025$:

$\cdot\dot{2}\dot{7}$, and $\cdot\dot{6}$: $\cdot072\dot{9}$ and $\cdot\dot{0}\dot{7}$: $\cdot\dot{1}4285\dot{7}$ and $\cdot\dot{6}\dot{3}$: $\cdot\dot{2}8571\dot{4}$ and $\dot{1}2\dot{3}$:

$\cdot1\dot{6}$ and $\cdot\dot{3}$: $\cdot9\dot{3}$ and $\cdot\dot{7}$: $\cdot4\dot{9}2\dot{5}$ and $\dot{7}\dot{5}$: $\cdot\dot{5}7142\dot{8}$ and $\cdot00\dot{5}\dot{4}$:

$\cdot3\dot{7}$ and $\cdot1\dot{4}\dot{8}$: $\cdot101\dot{3}$ and $\cdot000\dot{1}3\dot{2}$: $03\dot{4}2857\dot{1}$ and $\cdot00\dot{2}\dot{5}$:

$\cdot\dot{5}81\dot{9}$ and $\cdot015\dot{9}$: $\cdot41716\dot{2}6984\dot{1}$ and $\cdot12\dot{9}3\dot{1}$: $\cdot5\dot{7}1428\dot{5}$ and $\cdot6\dot{3}$:

$3\cdot\dot{1}4\dot{1}$ and $2\cdot31$: $13\cdot21\dot{0}\dot{1}$ and $1\cdot\dot{0}0\dot{1}$: $7\cdot00\dot{3}4\dot{2}$ and $\cdot0\dot{3}$:

$101\cdot50\dot{7}1428\dot{5}$ and $11\cdot6\dot{3}$: $3\cdot\dot{1}\dot{4}$ and $2\cdot006\dot{3}\dot{7}$:

$2\cdot61\dot{6}$, $\cdot0013\dot{2}$, $1\cdot04\dot{4}\dot{8}$ and $\cdot626\dot{3}\dot{9}$.

2. Shew that the product of two circulating decimals may produce a terminating decimal. Exemplify in the product of $\cdot2\dot{1}4285\dot{7}$ by $\cdot4\dot{6}$.

XIII.

1. Determine the quotients of the following decimals :—

$\cdot\dot{3}\dot{2}$ by $1\cdot6$, $\cdot16$, $\cdot016$, $\cdot0016$, 160, 1600, and 16000 respectively :

$\cdot\dot{7}\dot{2}$ by $\cdot\dot{3}$: $\cdot\dot{2}9\dot{7}$ by $\cdot\dot{2}\dot{7}$: $\cdot\dot{6}$ by $\cdot\dot{9}$: $\cdot\dot{5}\dot{4}$ by $\cdot\dot{4}\dot{5}$: $\cdot\dot{1}4285\dot{7}$ by $\cdot\dot{7}$:

$\cdot\dot{1}4285\dot{7}$ by $\dot{6}\dot{3}$: $\cdot\dot{6}\dot{3}$ by $\cdot\dot{1}4285\dot{7}$: $\cdot83$ by $\cdot1\dot{6}$: $\cdot041\dot{6}$ by $\cdot\dot{2}2\dot{5}$: $\cdot01\dot{3}$ by $\cdot000\dot{1}3\dot{2}$: $\cdot\dot{9}0\dot{1}$ by $\cdot\dot{1}0\dot{9}$:

$25\cdot\dot{2}\dot{5}$ by $2\cdot\dot{5}$: $\cdot00\dot{2}\dot{5}$ by $\cdot025$: $12\cdot9\dot{7}6923\dot{0}$ by $\cdot0\dot{8}5714\dot{2}$:

$\cdot4\cdot7\dot{5}4\dot{3}$ by $3\cdot4\dot{5}\dot{3}$: $1\cdot9\dot{5}6$ by $\cdot8\dot{3}\dot{6}$: $3\cdot42\dot{7}5\dot{3}$ by $2\cdot57\dot{3}2\dot{4}$.

2. Find the quotient arising from the product of $2\cdot\dot{6}1\dot{6}$ and $\cdot00132$, divided by the product of $1\cdot04\dot{4}\dot{8}$ and $6\cdot2639$.

XIV.

1. Shew that when the fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ are reduced to decimals, the periods of each will consist of the same digits. What explanation can be given?

2. In what sense can a finite fraction be said to be equal to an indefinite repeating decimal? How must the sign = be understood in the expression $\frac{35}{99}=\cdot353535$ . . . ad infinitum, in order that the expression may be satisfactory?

3. Shew that $\cdot03125=\frac{\cdot03}{1-\cdot04}=\frac{\cdot031}{1-\cdot008}=\frac{\cdot0312}{1-\cdot0016}=\frac{\cdot03125}{1-\cdot00000}=\frac{1}{32}$.
and $\cdot\dot{1}4285\dot{7}=\frac{\cdot14}{1-\cdot02}=\frac{\cdot142}{1-\cdot006}=\frac{1428}{1-\cdot0004}=\frac{\cdot14285}{1-\cdot00005}=\frac{\cdot142857}{1-\cdot000000}=\frac{1}{7}$.

4. If the denominator of a fraction be prime to 3, the period of the equivalent decimal is divisible by 9. Exemplify in $\frac{1}{13}$, $\frac{1}{17}$, $\frac{1}{19}$.

5. Convert $1\cdot44\dot{9}$ to an ordinary fraction, and then convert the fraction to a decimal, and explain the apparent discrepancy.

6. Find the first six decimal fractions of one, two, three, &c., places of figures which approximate to the fraction $\frac{13}{127}$, and shew that the successive differences of the fraction and the decimals decrease.

7. How many repeating and how many non-repeating figures will there be in the decimal equivalent to the fraction $\frac{98}{28672}$?

8. What must be forms of the numerator and denominator, that a fraction and its reciprocal may both be reducible to finite or repeating decimals?

9. If the number of figures in a recurring period of a decimal be one less than the denominator of the fraction, the sum of the figures in the period $= \frac{9}{2} \times$ number of figures. Give an example.

10. In every pure repeating decimal of an even number of repeating figures, the number composing the last half of the repeating period is less by unity than the arithmetic complement of the first half of the period. Exemplify this in reducing $\frac{11}{13}$ to a decimal.

11. In the reduction of a fraction to a pure repeating decimal having an even number of repeating figures, if the several remainders arising from the reduction of the fraction be arranged equally in two series, the sum of every two corresponding remainders will be equal to the denominator of the fraction. Exemplify this by reducing $\frac{4}{23}$ to a decimal.

CONCRETE DECIMALS.

XV.

1. Find the values of 1·0625 guineas; ·83229 of £1; ·0425 of £100; and ·35687 of a moidore.

2. Find the exact value of ·7365 of 6s. 8d.

3. Reduce ·165625 of one guinea to the decimal of a pound; and ·14526 of a pound to the decimal of a guinea.

4. Convert 8·775 shillings to the decimal of a moidore; and $\frac{3}{4}$d. to the decimal of a guinea.

5. Reduce 3s. 6d. to the decimal of £2 10s.; 3s. $11\frac{5}{7}$d. to the decimal of a moidore; and 14s. $10\frac{1}{7}$d. to the decimal of £1 4s. $3\frac{1}{2}$d.

6. Reduce £$\frac{\cdot 012}{\cdot 025}$ to the fraction of a farthing, and find what decimal of a farthing is one thousandth part of a pound.

7. Convert $\frac{3}{5}$ of one penny to the decimal of half-a-crown; $\frac{2}{9}$ of one guinea to the decimal of one pound; $\frac{2}{3}$ of 3s. 6d. to the decimal of $\frac{3}{5}$ of 6s. 8d.; $\frac{2}{3}$ of $\frac{5}{7}$ of one guinea to the decimal of £5; and £2 12s. 6d. to the decimal of £7 2s. $11\frac{1}{4}$d.

8. Reduce £2 12s. 6d. to the decimal of £1, and conversely £1 to the decimal £2 12s. 6d.

9. Find the value of ·047460975 of £10 13s. 4d., and ·00390625 of £20 10s. 6d.; and reverse the operations.

10. Reduce £693 15s. to the decimal of £750.

11. Explain the reason why any number of shillings may be expressed in the decimal of a pound by multiplying the shillings by 5, and marking off two places of decimals.

12. Shew that any sum of shillings, pence, and farthings may be converted into a decimal of a pound, correct to three places of decimals, by writing 1 in the first place for every two shillings, 50 in the second and third places for the odd shilling (if any), and 1 in the third place for every farthing additional, adding 1 extra if the number of pence exceeds sixpence. State the converse rule for turning the decimal of a pound into shillings, pence, and farthings.

13. If a decimal coinage were adopted, and the tenth, hundredth, and thousandth parts of a pound sterling called respectively florins, cents, and mils; what sum would 4 florins, 7 cents, 5 mils represent?

XVI.

1. Find the sum of £·1225, 1·225 crown, 12·25 shillings, 122·5 pence, in the decimal of a pound.

2. Add together £·0125, ·0125s., ·0125d., and ·0125 farthing.

3. Add together 12·125 pounds, 17·3025 shillings, and 9·75 pence, and express the result in the decimal of £10.

4. Find the values of ·971875 of one pound, of one shilling, of one penny, and of one farthing, and their sum.

5. Express the sum of £5·125, 3·625s., 9·75d., and 3·375 farthings, in the decimal of a crown.

6. Subtract $\frac{3}{8}$ of five crowns from £1·59375.

7. Which is the greater, ·36 of a guinea or $\cdot\dot{3}\dot{6}$ of a pound?

8. Divide £12 5s. 6d. by ·000625.

9. Divide £628125 by 2·778625 of 6s. 8d., and state the nature of the quotient.

10. By what decimal of one farthing does ·0009 of one shilling exceed ·00003 of one pound?

11. Shew that ·025 of £1 is double of ·05 of £1 5s.

XVII.

1. Determine the exact values of the following concrete repeating decimals:—

£$\cdot 012\dot{3}$; £$1\cdot 3\dot{9}$; £$3\cdot 68\dot{4}2\dot{2}$; $\cdot 23\dot{4}\dot{5}$ of £6 17s. 6d.; $\cdot 810\dot{7}\dot{4}$ of $7\frac{1}{2}$ guineas; $\cdot 378\dot{9}3\dot{4}$ of a crown.

2. What decimal of £20 12s. 6d. is £7 17s. 6d.?

3. Find the exact sum of £$2\cdot 15\dot{6}\dot{3}$, $12\cdot\dot{5}\dot{4}$s., and $9\cdot\dot{3}$d.

4. What is the sum and difference of £$3\cdot 4\dot{5}\dot{2}$ and $2\cdot 1\dot{7}\dot{2}$ guineas?

5. What is the difference of $\cdot 17\dot{3}\dot{6}$ of £1 11s. 6d. and $\cdot 231\dot{2}$ of a guinea?

6. Subtract $\frac{17}{29}$ of a pound from $\frac{23}{27}$ of a guinea, and express the result as a decimal of a pound.

7. Multiply £$\cdot 12\dot{3}\dot{5}$ by $3\cdot 2\dot{5}$ and divide £$\cdot 12\dot{4}\dot{9}$ by 1·27s., and state the nature of the results in each case.

8. Between what numbers of farthings does the value of £$151\dot{6}\dot{3}$ lie?

XVIII.

1. If the value of a rupee be 2s. 4d., express £7·5642 in rupees and decimal parts of a rupee.

2. If £1 sterling be worth 12 florins, and also worth 25·56 francs, how many francs are there in one florin?

3. An American dollar is worth 4s. $3\frac{3}{5}$d. or 5·42 francs; what is the least sum which can be paid in either shillings, dollars, or francs?

4. If 11·75 florins of Amsterdam be worth 13·5 marks of Hamburgh, and 120 florins be worth 25 lire of Genoa, and 5 lire be worth 800 Portuguese rees, and 1000 rees be worth 54 pence; what advantage or disadvantage will there be between sending £500 direct from London to Hamburgh at an exchange of 13·6875 marks for one pound sterling, or sending it by way of Lisbon, Genoa, and Amsterdam?

XIX.

1. Reduce $\frac{23}{15}$ of an ounce to the decimal of a pound Troy; and reverse the operation.

2. Find the exact values of the following concrete decimals: ·379 of an ounce, and $\cdot 9\dot{5}\dot{4}$ of a pound Troy.

3. If the unit of Troy weight called the pennyweight counted 14·25 grains instead of 24, how many grains would make the pound Troy?

4. What number will represent 116·0435 grains when 4·0015 grains is the unit of weight?

5. If the Attic silver drachma be taken at $9\frac{3}{4}$d. when its exact value is 9·72 pence, what difference does this make in the talent of silver, if a talent consisted of 9,000 drachmæ?

6. If $25\frac{1}{2}$ francs be worth £1, and gold be worth £3 17s. $10\frac{1}{2}$d. per ounce, how much pure gold could be bought for a franc?

XX.

1. If silver be worth 5s. 6d. an ounce, and pure gold be worth £4 5s. an ounce, what should be the weight of a fifteen-shilling piece containing 92·5 per cent. of pure gold and the rest silver?

2. The value of one ounce Troy of English standard gold is £3 17s. $10\frac{1}{2}$d., 31·1 French grammes are equal in weight to one ounce Troy, and French standard gold is $\frac{60}{61}$ of the value of English standard gold. If one gramme of French gold be worth 3·1 francs, what is the worth of one pound sterling in francs?

3. One kilogramme of French standard gold $\frac{9}{10}$ fine is coined into 155 Napoleons. One pound of English standard gold $\frac{11}{12}$ fine is coined into 1,869 sovereigns. If a sovereign be reckoned as equivalent to 25·22 francs, find the equivalent of a kilogramme in grains.

4. If a sovereign weigh 5 dwts. $3\frac{1}{4}$ grs., one part out of 12 being alloy and the rest pure gold; find what fraction of a cubic inch the

gold constitutes, having given that a cubic inch of water weighs 252·458 grains, and that gold is 19·362 times as heavy as water.

XXI.

1. Reduce 11 lbs. 11 oz. 11 drams to the decimal of a hundredweight; and reverse the operation.

2. What decimal of 5 cwt. 1 qr. 17 lbs. 3 oz. is 4 cwt. 23 lbs. 3 oz. Avoirdupois?

3. Determine the exact value of $\cdot 03\dot{4}5\dot{6}$ of one ton and $\cdot\dot{1}\dot{7}$ of one hundredweight.

4. If the unit be ·0064 ounce Avoirdupois, what is the smallest number of such units which can be represented by an integer?

5. Add ·24 to 5·03, subtract from this sum 3·002, multiply the difference by 40. Divide this product by ·007; and supposing the original unity to have been a hundredweight, find the value of the resulting decimal in tons, cwts., &c.

XXII.

1. Express a pound Troy as the decimal of a pound Avoirdupois; and conversely, express a pound Avoirdupois as the decimal of a pound Troy.

2. If the Roman pound contained 63460·6 French grains, find how many grains Troy are contained in the Roman pound if the French grain is equal to ·8202 of the grain Troy.

3. If 40,443,495 sovereigns weigh 10,386,772 ounces, and 150,291,456 pennies weigh 1,442 tons 12 cwt., find to three places of decimals the weight of a sovereign and of a penny in grains.

4. If gold be sold to the mint at £3 17s. 9d. per ounce, of which 11 parts are pure gold and one part alloy, and be coined into sovereigns; what is the difference of the mint profit on 10,000 sovereigns, if the alloy be silver worth 5s. 2d. an ounce Troy, or copper worth 1s. 9d. per pound Avoirdupois, supposing each sovereign to weigh 123·247 grains?

5. At the English mint 1869 sovereigns are coined from 40 lbs. Troy of standard gold $\frac{11}{12}$ fine. At the French mint, 155 Napoleons are coined from one kilogramme (2·2 pounds Avoirdupois) of standard gold $\frac{9}{10}$ fine. A Napoleon being equivalent to 20 francs, find how many francs are equivalent to an English sovereign.

XXIII.

1. Find the value of ·7854 mile in yards, feet, and inches.

2. Find the exact value of $\cdot 012\dot{3}$ of $2\frac{1}{2}$ miles; and reverse the process.

3. Express in the fraction of a foot, the remainder after ·012 of a yard has been subtracted as often as it is possible from 1·087 yard.

4. How many times can 2·372 feet be subtracted from 42·8731 yards, and what is the length of the remainder?

5. If in a yard measure there was an error of one-tenth of an inch, what would be whole error in measuring 5 miles and 345 yards? What is the difference when the error is in excess or defect?

XXIV.

1. What is the difference of the Roman and the English mile, the former of which consisted of 5,000 Roman feet, and each foot ·9708 of an English foot?

2. If 24 Roman feet are equal to 25 Greek feet, find the length of the Roman foot, if the length of the Greek foot be 12·137 English inches.

3. Pliny (Lib. ii. 21, &c.) states the stadium to be equal to 625 Roman feet, and 8 stadia of 600 Greek feet are equal to a mile of 5,000 Roman feet. If the Roman foot was 11·6496 English inches, what was the length of the Greek foot?

4. The length and breadth of the upper step of the basement of the Parthenon at Athens, by admeasurement, was found to be in English measure 227 ft. 7·05 inches, and 101 ft. 1·7 inches respectively; and admitting the term hecatompedon was applied to it on account of its dimensions, show that the value of the Greek foot as determined from the length and from the breadth was 12·138 and 12·137 inches respectively.

XXV.

1. If the French metre be 39·3708 inches, what fraction of a metre is a yard?

2. Express a degree, sixty-nine miles and a half, in French metres, 32 metres being equal to 34·99625 yards.

3. Find the value of the metre of France in terms of the foot of Cremona, if 48 Cremonese feet be equal to 56 English feet, and if the metre be equal to 39·371 English inches.

4. The length of a seconds pendulum is 39·1393 inches; what is that in yards, feet, inches, and fractions of an inch?

5. How many crowns and how many half-crowns, whose diameters are respectively ·81 and ·666 of an inch, may be placed in two rows close together so that each row may make a yard in length?

XXVI.

1. The Mont Cenis Tunnel is 12,224 metres long; express its length in miles and yards, one metre being 39·37079 inches.

2. Ten yards make 9·14382 French metres, and a metre is the ten-millionth part of the earth's quadrant; find the circumference of the earth in miles and furlongs.

3. The mean distance of the earth from the sun is 15,287,813

French metres; what is the distance in English miles, if the metre be equal to 3·2809 English feet?

4. If the mean length of a degree of latitude be 365,000 feet, find the length of the French metre, which is one ten-millionth part of the distance from the pole of the earth to the equator, in feet and inches.

5. If the mean distance of the moon from the earth be 59·97 times the earth's radius, and if this radius be 3962·824 miles; find the distance of the moon from the earth in miles, yards, feet, and inches.

XXVII.

1. The base of the great pyramid at Gizeh is a square, each side of which is 763·4 feet; find the area in acres of the ground on which the pyramid stands.

2. If the length of a metre be 39·37 inches, find the number of square feet and the number of cubic feet in that square and cube whose side is a metre.

3. How many cubical packages, the edge of each measuring 3·03 inches, can be packed in a box whose content is 1030·301 cubic inches?

4. A wire 99 miles long, and the area of whose section is ·01 inch, is drawn out into a finer wire 100 miles long; shew that the area of its section is ·0001 inch less than before.

5. If the length of the cubit be 24·883 English inches, what was the cubical space of Noah's ark, its length, breadth, and height being respectively 300, 50, and 30 cubits respectively?

XXVIII.

1. If the value of gold be £3 10s. 10½d. an ounce, and a cubic inch of gold weigh 10 ounces, what quantity of gold would be required to gild a dome whose surface was 5,000 square feet, and the thickness of the gold ·0002 of an inch thick?

2. The weight of a cubic foot of water being 1000 ounces, find the weight of a rectangular block of gold 8 inches in length, 2 in thickness, and 3 in breadth; the weight of a mass of gold being 19·25 times the weight of an equal bulk of water.

3. A cubic foot of iron weighs 7·8 times as much as a cubic foot of water; find the weight of a block of iron 10·41 feet long, 2·58 feet broad, and 3 feet thick; supposing a cubic foot water to weigh 1,000 ounces.

XXIX.

1. Find a multiplier for converting cubic inches into gallons.

2. If the weight of one cubic foot of water be 62·35 pounds Avoirdupois, find the error in calculating the weight of 1000 cubic feet on each of the following approximate assumptions:—

1. That one cubic inch weighs 252·5 grains.
2. That one cubic foot weighs 1,000 ounces Avoirdupois.
3. That one cubic fathom weighs 6 tons.

3. If a gallon contain 277·274 cubic inches, and a cubic foot of water weigh 1000 ounces, what quantity and weight of water will fill a rectangular cistern 5 feet by $3\frac{1}{2}$ feet wide, and 2 feet 9 inches deep?

4. A gallon of water contains 277·274 cubic inches, and weighs 10 pounds Avoirdupois; find how many tons of water are in a dock whose area is an acre and depth 10 feet.

5. A gallon of water weighs 146·170 ounces Troy. Find the weight of a cubic foot of water in ounces Avoirdupois, if a pint contains 34·66 cubic inches.

6. If 144 pounds Avoirdupois are equal in weight to 175 pounds Troy, how many pounds Troy are contained in the imperial gallon of 272·274 cubic inches, if the weight be 10 pounds Avoirdupois?

7. A gallon contains 277·274 cubic inches, and a cubic foot of water weighs 997·784 ounces; find the weight of a pint of water in pounds Avoirdupois to three places of decimals.

XXX.

1. If water expand ten per cent. when it turns into ice, how much does ice contract when it turns into water?

2. Find the number of gallons of water which pass in 10 minutes under a bridge 17 feet 8 inches wide, the stream being 10 feet 11 inches deep, and its velocity 8 miles an hour, supposing a gallon contains 277·274 cubic inches.

3. If the annual average of the rainfall in a certain district, in the periods of summer, autumn, and winter, be 2·84, 30·3, and 50·6 per cent. respectively, how much per cent. falls in the spring?

4. If the lineal expansion by heat of a piece of metal one unit in length be ·01, show that the expansions in surface and volume of the corresponding units are ·0201 and ·030301 respectively.

5. If the atmosphere were of the same density throughout as it is at the level of the sea, its height would be 26253 feet; find the weight of the air which surrounds the earth, having given that 875·1 cubic feet of air weigh as much as ·1 cubic foot of mercury, and 30 cubic inches of mercury weigh 14·7 lbs.: supposing the earth a sphere of 8,000 miles in diameter, and having given the content of a sphere equal to two-thirds of its circumscribing cylinder, and the area of a circle equal to 3·1415926 times the square of the radius.

6. If a room be 10 feet high, 50 feet long, and 35 feet broad, find the weight of air contained in it, supposing that 100 cubic inches weigh 32·698 grains at the temperature of 32° Fahrenheit's thermometer.

7. How high will the water barometer be if the mercurial barometer stand at 29·9218 inches, having given that one cubic inch of mercury weighs 13·5962 cubic inches of water?

XXXI.

1. The inscription on the congius of Vespasian preserved at Dresden, states that it contains 10 Roman pounds. When this measure was filled with water and carefully weighed, the weight of the water was found to be 63460·6 French grains; find how many grains Troy are contained in one Roman pound, if the French grain is equal to ·8202 of the English grain.

2. By the statute of William III. the Winchester bushel is declared to be round with a plain bottom, $18\frac{1}{2}$ inches wide throughout, and 8 inches deep. How many cubic inches is the content, the area of a circle being equal to 3·14159 times the square of the radius?

3. If the Roman modius contained three congii, and the congius of Vespasian preserved at Dresden contains ten Roman pounds of water, how many English gallons make a modius?

4. Pliny states that a modius of the corn of Clusium weighed twenty-six Roman pounds. Suppose that a congius and a gallon are filled with the same grain, and that the former weighs as many Roman pounds as the latter weighs pounds Avoirdupois, and if a modius be three congii, what would be the weight of an imperial bushel of the corn of Clusium?

XXXII.

1. Reduce 23° 15′ 11″ to the decimal of a degree in circular measure.

2. Express 13° 17′ 8″ as the decimal of a right angle.

3. Reduce $\frac{180}{3 \cdot 14159}$ of a degree to degrees, minutes, and seconds.

4. If 90 English degrees correspond to 100 French degrees, how many French degrees are there in 36·45 English?

XXXIII.

1. What decimal of a day is 13 hours 12 minutes 10 seconds?

2. What decimal of a day of twelve hours is 5 hours 48 minutes 49·7 seconds? Find the value of ·85714 of a calendar month.

3. Reduce 3 days 5 hours 48 minutes 57 seconds to the decimal of a Julian year.

4. Divide 3 weeks 4 days 5 hours 6 minutes 7 seconds by 5 days 6 minutes 40 seconds, expressing the result as a decimal and stating its nature.

5. Divide ·375 of a calendar month by ·7854 of a week, and state the nature of the quotient.

6. Find the sum of $\cdot 12\dot{3}$ of a Julian year, ·565 of a week, ·0125 of an hour, in days.

7. Find the decimal of a week which differs from a day by less than the millionth part of a week.

8. For what unit of time can 108·8 and 8·28 hours be represented by numbers prime to each other?

XXXIV.

1. The sun appears to a person on the earth to move through 360° in 365·24 days; shew it moves through 59′ 8″ ·351 . . . in one day.

2. Assuming the length of the Julian year at 365 days 6 hours, find the decimal which represents one hour to 10 places of figures.

3. Find what time in hours, minutes, and seconds corresponds to 23° 27′ 53″ in the revolution of the earth on its axis; and conversely.

4. If the circumference of a circle be 3·14159 times the diameter, find at what rate per hour a body moves at the equator by the rotatory motion of the earth, supposing the equatorial diameter of the earth to be 7925·648 miles.

5. The longitude of Calcutta is 88° 20′ east, and of Barbados 59° 50′ west of London. Convert these longitudes into time, and shew how much the clocks of London are before or behind those of Barbados and Calcutta; also how much the clocks of Calcutta are before or behind those of Barbados.

6. A lunar month is 29·530588 days. How many lunar months are there in 19 solar years, calculated at 365·242264 days?

7. There was a full moon on the 26th June, 1858, at 9.13 a.m. The interval between successive full moons has since been on the average 29 days 12 hours 47½ minutes; how many full moons have there been between the 26th June, 1858, and 26th June, 1875?

XXXV.

1. Describe the Julian and Gregorian adjustments of the calendar, and explain why only every 400th year is a leap year.

2. How many days elapsed between the epoch of the correction of the calendar by Julius Cæsar, January 1, B.C. 45, and September 14, A.D. 1752, the date of the adoption of the Gregorian correction in England?

3. The solar year contains 365·242218 days, and the average Julian year 365·25 days. In the year 1582 Pope Gregory corrected the Julian calendar; how many days in round numbers did he add or omit in that year to make it coincide with solar time?

4. The length of the tropical year being 365·242264 days, compare the accuracy of the Gregorian intercalation with that of the Persian, in which 8 days were intercalated in 33 years.

5. If the year be divided into eighteen months of twenty days each, and five intercalary days be added at the end of each year, and also 12½ intercalary days at the end of each cycle of 52 such years, in how long a time will the error in reckoning amount to one day, the true length of a year being 365·242264 days?

6. The true length of the year is 365·242264 days, and the calendar as corrected by Julius Cæsar supposed it to be 365¼ days; in how many years will the error caused by this discrepancy amount to one week?

RESULTS, HINTS, ETC., FOR THE EXERCISES.

Abstract Decimals.

I.

1. Art. 1. 2. $321{\cdot}123 = 300 + 20 + 1 + \frac{1}{10} + \frac{2}{100} + \frac{3}{100}$, which expressed in words is three hundred and twenty-one, together with one-tenth, two-hundredths, and three-thousandths. 3. Art. 2. 5. Art. 4, note, p. 3.

II.

(1) 725·2251. (2) 21·2534. (3) 13874·763903.
(4) 106·32275. (5) ·0095.

2. The following are the decimals which can be formed with the four figures:

·0379
·0397
·0739
·0793
·0973
·0937

·3079
·3097
·3709
·3790
·3970
·3907

·7039
·7093
·7390
·7309
·7930
·7903

·9037
·9073
·9307
·9370
·9730
·9703

The sum of these decimals is 12·6654.

III.

The following are the differences of the decimals:
·018 : ·001 : ·096 : 17·28659 : ·179973 : 233·90001 : ·099 : 785·999307 : ·00732421375.

IV.

1. Art. 6. 3. 9·25 : ·2 : 15 : ·842 : 1000 : ·1 : 758642·5 : 15·525 : 15·625 : ·00003738028 : ·000001 : 82·75139486 : 33·57845799 : ·000365625 : 56·173002 : ·0000000289536. 4. 2·2939153468.

V.

1. ·1 : ·3 : ·003 : ·005 : ·00011 : ·000000001 : ·0079 : 1·25 : ·06 : ·005 : 170·01 : 18·45 : 10·003.

2. 2 : 12 : 525 : 37 : 46 : 618 : 873 : 151.

3. 40 : 3680 : 400000 : 5000 : 70 : 17750000 : 605000000 : 1600.

4. ·0005 and 2000 : 1000000 and ·000001 : 2·5 and ·4 : ·4 and 2·5 : 4 and 2·5.

5. Art. 8. 6. Art. 7. The value of the remainder is estimated after the division from the decimal point in the dividend to the last figure used in the division.

7. The quotient of the first example is 20495·1856946, and the value of the remainder is ·000000000258. 8. 1·002... 9. ·01. 10. 499·998. 11. 14·9816. 12. ·0000005.

VI.

1. If ·625625 be a divisor, the equivalent multiplier is $\frac{1}{\cdot 625625}$ or $\frac{1001}{1600}$. 2. The multiplier is 15625. 3. 625. 4. 263 times, with a remainder ·0027, which is one-third of ·0087. 5. 3·97 is the quotient, and ·00035 is the remainder. 6. ·65. 7. The least common multiple.

VII.

1. Art. 10. 2. Art. 10. 4. $\frac{355}{113} = 3{\cdot}1415964$, the error is ·0000038.

VIII.

1. 3·9515026. 2. ·7604602. 3. 1·7177815. 4. ·4993585. 5. ·00000000096098035. 6. ·0031209.

IX.

1. Arts. 11—14. 2. Art. 11. 5. Art. 10, note, p. 7.

X.

1. Art. 15. The L.C.M. of 4 and 6 is 12, which is the number of repeating decimals in the sum and difference of two repeating decimals of four and six places respectively.

2. $123\cdot\dot{6}624\dot{2}$: $1\cdot\dot{0}2431324213\dot{2}$: $38\cdot1848\dot{9}3884\dot{7}$: $6\cdot0369$: $315\cdot2\dot{0}41283433362\dot{6}$: $263\cdot32\dot{8}4730\dot{3}$: $473\cdot29\dot{1}4634076020\dot{2}$: $18\cdot\dot{6}$: $215\cdot043283428\dot{2}3\dot{4}$.

3. 10.

XI.

1. $\cdot8\dot{9}\dot{3}=\frac{59}{66}$: $\dot{3}7\dot{0}=\frac{10}{27}$: $\cdot\dot{4}2742\dot{9}=\frac{427429}{999999}$: $\cdot570\dot{6}\dot{5}=\frac{11299}{19800}$: $\cdot65\dot{9}7488\dot{3}$: $\cdot010\dot{7}1428\dot{5}$: $11\cdot11\dot{2}27\dot{0}$: $\cdot212\dot{4}\dot{9}$: $\cdot0\dot{0}0191\dot{1}$; $\cdot25\dot{3}898\dot{8}$.

2. $\frac{5}{7}-\frac{17}{64}=\frac{201}{448}$.

XII.

1. $3\frac{323}{750}$: $356\frac{21}{37}$: $\frac{400909}{499500}$: $\frac{41716228125}{399999960000000}$, reduce to lowest terms : $\frac{5}{27}$: $\frac{511}{99000}$: $\frac{1}{11}$: $\frac{86}{2331}$: $\frac{1}{18}$: $\frac{98}{135}$: $\frac{24605}{65934}$: $\frac{6}{1925}$: $\frac{833}{14850}$: $\frac{209}{15609375}$: $\frac{1}{11550}$: $\frac{646}{69375}$: $\frac{479049451919}{887999911200000}$: $\frac{38}{105}$: $7\frac{1421}{5550}$: $13\frac{22027}{98901}$: $\frac{38869}{166500}$: $1166\frac{813}{1400}$: $6\frac{2968241}{9801000}$: $\frac{157}{10}\times\frac{119}{90000}\times\frac{431}{2475}\times\frac{20671}{33000}$.

2. The product is ·1 a terminating decimal.

XIII.

1. $\frac{2}{9}$: $2\frac{2}{9}$: $22\frac{2}{9}$: $222\frac{2}{9}$: $\frac{1}{450}$: $\frac{1}{4500}$: $\frac{1}{45000}$: $2\frac{2}{11}$: $1\frac{10}{111}$: $\frac{2}{3}$: $1\frac{1}{5}$: $\frac{9}{49}$: $4\frac{5}{11}$: $4\frac{49}{50}$: $\frac{37}{200}$: $101\frac{17}{33}$: $8\frac{29}{109}$: $9\frac{223}{253}$: $\frac{10}{99}$: $151\frac{31}{78}$: $1\frac{47649}{126503}$: $2\frac{937}{2760}$: $1\frac{28448}{85689}$.

2. The quotient is $\frac{1317\times132}{5219\times62639}$.

XIV.

2. Any finite fraction can only be said to be equal or equivalent to the infinite repeating decimal, as the limit of the value which the decimal can never exceed. It may easily be shewn that the more figures of the decimal are taken, the larger the decimal becomes, and will continue to approach in actual value to the fraction, but within a difference less than can be assigned by any fraction whatever.

5. Art. 13. 6. Art. 9, note 2, p. 6.

7. This fraction having the factor 7 in the denominator, is apparently one which will produce a repeating decimal, but when the fraction is reduced to its lowest term, the denominator consists of factors each equal to 2. Art. 10.

8. Both the numerator and the denominator must contain one or more factors different from 2 and 5. Art. 10.

9. Take $\frac{6}{7}$ as an example, $\frac{6}{7}=\cdot\dot{8}7514\dot{2}$. The number of repeating figures is 6, and the sum of the digits of the period is 27, and $27=\frac{9}{2}\times6$.

10. Here $\frac{11}{13}=\cdot\dot{8}4615\dot{3}$: and the A. Complement of 154 is $1000-154=846$.

11. When $\frac{4}{23}$ is reduced to a decimal, the repeating figures in the decimal are 22, and the first eleven and the second eleven remainders are respectively—

17, 9, 21, 3, 7, 1, 10, 8, 11, 18, 19,
6, 14, 2, 20, 16, 22, 13, 15, 12, 5, 4.

CONCRETE DECIMALS.

XV.

1. £1 2s. $3\frac{3}{4}$d.: 16s. $7\frac{1}{2}$d. ·9984 : £4 5s.: £4 16s. $4\frac{1}{4}$d. ·0352. 2. 9s. $9\frac{3}{4}$d. $\cdot3\dot{6}$.
3. One the reverse of the other. 4. ·325 of 1 moidore, and $\cdot0029\dot{7}6190\dot{4}$ of one

guinea. 5. ·07 of £2 10s.: ·$\dot{1}$·1472663... of 1 moidore. The fraction $\frac{2\times29\times43}{7\times11\times53}$ when reduced to a decimal, will be the required decimal of £1 4s. 3½d. 6. $\frac{2304}{5}$ of one farthing : $\frac{24}{25}$ of one farthing. 7. ·02 of half-a-crown : ·2$\dot{3}$ of £1: ·58$\dot{3}$ of $\frac{3}{5}$ of 6s. 8d.: ·1 of £5 : ·36729... of £7 2s. 11¼d. 8. The one is simply the reverse of the other. The value of ·017460975 of £10 13s. 4d. is 1·0025007$\dot{3}$s. 10. ·925 of £750.

13. 1 florin = $\frac{1}{10}$ of £1 : 1 cent = $\frac{12}{5}$ of 1 penny : 1 mil = $\frac{6}{25}$ of 1 penny.

	s.	d.
4 florins =	8	0
7 cents =	1	4$\frac{4}{5}$
5 mils =	0	1$\frac{1}{5}$
	9	6

It may be questioned whether the advantages of the decimal division of money, weights, and measures could compensate for all the inconveniences which would arise on a departure from the established usage of a country. It has been truly remarked that "decimal arithmetic is a contrivance of man for computing numbers, and not a property of time, space, or matter. Nature has no partialities for the number ten, and the attempt to shackle her freedom with it, will for ever prove abortive."

XVI.

1. £1·5516. 2. £·0131901041$\dot{6}$. 3. £1·303075. 4. £1 0s. 6·12734375d. 5. 21·4015625 crowns. 6. £1 2s. 6d. 7. $\dot{3}\dot{6}$ of £1. 8. ·000625 as a divisor is equivalent to 1600 as a multiplier. 9. Express the dividend and divisor in the same units; the quotient will indicate how many times the latter sum is contained in the former. 10. ·0144 of a farthing.

XVII.

1. 2$\frac{21}{25}$d.: £1 8s.: £3 13s. 8$\frac{346}{1665}$d.: £1 12s. 3d.: £6 7s. 8$\frac{86}{275}$d.: 1s. 10$\frac{6128}{8325}$d. 2. £7 17s. 6d. = $\frac{21}{25}$ = ·38$\dot{1}$ of £20 12s. 6d. 3. £2 16s. 5$\frac{3072}{8085}$d. 4. ·319731... of £1. 5. 7¼d. $\frac{131}{275}$. 6. £4946... 7. £·00040217...: 1·2$\dot{7}$s. is contained 1·9557... times in £124$\dot{9}$. 8. Between 145 and 146 farthings.

XVIII.

1. 64·836 rupees. 2. 2·13 francs. 3. £10 15s. 4. The disadvantage will appear from the fact that the direct remittance gives 68437½ marks, but by Lisbon, Genoa, and Amsterdam, 76595$\frac{35}{47}$ marks.

XIX.

1. $\frac{23}{15}$ ounce = ·12$\dot{7}$lb. Troy. 2. 7dwts. 14$\frac{2}{5}$ grains : 11oz. 9dwts. 2$\frac{2}{11}$grs. 3. 3220 grains. 4. 29. 5. £1 2s. 6d. 6. 4$\frac{13708}{31773}$ grains.

XX.

1. Since 1oz. of silver is worth 5s. 6d., $\frac{45}{11}$dwts. of silver is worth 15s. And since 1oz. of gold is worth £4 5s., $\frac{111}{34}$dwts. of gold is worth 15s. But 92·5 + 7·5 = 100, and $\frac{92\cdot5}{100}+\frac{7\cdot5}{100}=1$, or $\frac{37}{40}+\frac{3}{40}=1$, that is, $\frac{37}{40}$ of the weight is gold, and $\frac{3}{40}$ silver; which parts together will be found to be 7$\frac{13}{14}$dwts.

2. £1 sterling weighs $\frac{160}{623}$ of an ounce Troy of English standard gold. This is to be converted in French standard gold, then this French gold into its equivalent value in francs.

XXI.

1. ·1047675... of 1cwt. 2. The fraction is $\frac{7533}{9683}$, which when reduced to a

decimal answers the question. 3. $78\frac{42}{10663}$ of 1 ton: 9lbs. $2\frac{10}{99}$oz. 4. ·0064 is contained 15625 times in 1,000,000. 5. 10 tons 10cwt. 89lbs. 9oz. $9\frac{3}{5}$drs.

XXII.

1. Reduce $\frac{84}{175}$ and $\frac{175}{84}$ to decimals. 2. The Roman pound contained 52050·38412 grains Troy. 3. Divide the number of grains in 13380772 ounces Troy by 40443495; and the number of grains in 1442 tons 12 cwt. Avoirdupois by 150291456, each to 3 places of decimals, the quotients will give the weights of each in grains.

XXIII

1. 1382yds. 0ft. $10\frac{118}{125}$ inches. 2. $54\frac{4}{15}$yds. 3. The remainder is $\frac{21}{1000}$ of 1 foot. 4. 54 times with a remainder of $7\frac{1439}{2500}$ inches. 5. The error of one-tenth of an inch in one yard will cause an error of $914\frac{1}{2}$ inches in 9145 yards. If 9145 yards be divided by $35\frac{9}{10}$, and by $36\frac{1}{10}$ respectively, the difference can be found when the error is in defect and excess.

XXIV.

1. The English mile exceeds the Roman mile by 426 English feet. 2. 12·642 English inches. 3. 12·135 English inches.

XXV.

The French metre is $\frac{32809}{30000}$ of the English yard. 2. $111847\frac{11581}{27997}$ French metres. 3. 1 metre = 2·812 Cremonese feet. 4. 1yd. 0ft. $3\frac{1393}{10000}$ inches. 5. 44 crowns, and ·36 of an inch over, and 54 half-crowns, and ·036 of an inch over.

XXVI.

1. 7 miles 1048·5728 yards. 2. The circumference of the earth is 40 million metres. 3. Convert the metres into feet, and the feet into miles. 4. These data give the length of the metre 39·42 inches. 5. 237650 miles, 977yds. 10·512 inches.

XXVII.

1. 13 acres 1833yds. 2ft. 80·64 inches. 2. 35·2232 cubic feet, and 10·7388 square feet. 3. 37. 5. 44501·49841 cubic yards.

XXVIII.

1. 1440 ounces of gold at the cost of £5103. 2. $1090\frac{5}{18}$ ounces. 3. 11 tons, 0cwt. 14lbs. 8·52oz.

XXIX.

1. One gallon contains 272·274 cubic inches; and 272·274, or $\frac{272274}{1000}$ as a divisor, will be equivalent to $\frac{1000}{272274}$, or $\frac{500}{136137}$ as a multiplier.

2. If 1 cubic foot of water = 62·35lbs., then 1000 cubic feet = 1000 × 62·35 = 62350lbs.

$$\text{On the first assumption 1000 cubic feet} = \frac{1000 \times 1728 \times 252{\cdot}5}{7000} = 62331\tfrac{3}{7}\text{lbs.}$$

$$\text{On the second} \quad ,, \quad ,, \quad = \frac{1000 \times 1000}{16} = 62500\text{lbs.}$$

$$\text{On the third} \quad ,, \quad ,, \quad = \frac{1000 \times 6 \times 20 \times 112}{216} = 62222\tfrac{2}{9}\text{lbs.}$$

3. The cistern contains $48\frac{1}{8}$ cubic feet of water, weighing $1320\frac{5}{10}$lbs., or $303\frac{23039}{138637}$ gallons. 4. 2423 tons 9cwt. $39\frac{23021}{138637}$lbs. 5. $999\frac{25277}{60655}$lbs. 6. $12\frac{11}{72}$lbs. Troy. 7. 19·940 ounces.

XXX.

1. Since water on freezing expands in volume 10 per cent., or 10 parts in 100, or 1 part in 10; 10 parts of water become 11 parts of ice.

Hence conversely, when 11 parts of ice become 10 parts of water, the ice is contracted by one-eleventh part of its volume.

2. 851208$\frac{12504}{138637}$ gallons. 3. 16·26 per cent.

5. If 3·141592659 times the square of the radius be taken for the area of a circle, and 5 miles be employed instead of 26253 feet, the volume of the atmosphere will be found to be the difference of two spheres whose radii are respectively 4000 and 4005 miles, and equal to 100656681 cubic miles nearly. Next finding what weight of mercury is equivalent to a cubic mile of air, the whole weight of the atmosphere may be determined according to the suppositions made.

6. 12·61208cwt.

XXXI.

1. 5205·384... grains Troy. 2. 2150·428... cubic inches. 3. 2·225... gallons. 4. 69$\frac{1}{3}$lbs. Avoirdupois.

XXXII.

1. 23·25305 degrees. 2. $\cdot 147\dot{6}1728395\dot{0}$ of a right angle. 3. 57° 17′ 44·6″. 4. 40·5 French degrees.

XXXIII.

1. $\cdot 550115\dot{7}4\dot{0}$ of 1 day. 2. ·4844421... of a day of 12 hours: 25 days 17hrs. 8min. 26·88sec. 3. ·6634534... of a Julian year. 4. 5·0376... an abstract number. 5. The decimals of the month and the week must first be reduced to days before performing the operation. 6. $49\cdot 0030208\dot{3}$ days. 7. $\cdot 142850\dot{1}4285\dot{7}$ of a week is the exact difference: any less decimal will fulfil the condition. 8. 4 hours.

XXXIV.

2. 1 hour = ·0001140791... of a Julian year. 3. 1 hour, 33min. 51$\frac{8}{15}$sec. 4. 1037·133... miles an hour.

5. 88° 20′ longitude correspond to 5hrs. 52$\frac{1}{3}$min., and 59° 50′ to 3hrs. 59$\frac{1}{3}$min. Suppose the time to be noon at London, then the clocks of Calcutta are 3hrs. 52$\frac{1}{3}$min. after 12, and the clocks of Barbadoes are 3hrs. 59$\frac{1}{3}$min. before 12: or the clocks of Calcutta indicate 7min. 40sec. past 6 o'clock p.m., and those of Barbadoes 40sec. to 8 o'clock a.m. The clocks of Calcutta are 9hrs. 51$\frac{2}{3}$min. behind the clocks of Barbadoes.

6. Divide the number of days in 19 years by the number of days in one lunar month.

7. Divide the whole period by the interval between two successive full moons.

XXXV.

1. See "On the Divisions and Measures of Time," pp. 14, 17. 2. The product of the correct number of days in one year, and the number of years in the interval, will give the required number. 3. The error in one year is ·007736 of a day, which in 1257 years amounted to 9·724152 days, or 10 days nearly, the number of days omitted in 1582 by Pope Gregory in his adjustment of the calendar. 4. It may be shewn that the Gregorian intercalation is more accurate than the Persian. 5. The error in 52 years is ·34728 of 1 day, and this error amounts to one day in 52×2·8795 = 149·734, or nearly 150 years.

ELEMENTARY ARITHMETIC,

WITH BRIEF NOTICES OF ITS HISTORY.

SECTION XI.

PROPORTION.

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RATIO AND PROPORTION.

Art. 1. Two numbers or two magnitudes of the same kind may be compared in two ways; by considering, first, how much one of them is greater than the other; and, secondly, how many times one contains, or is contained in, the other. The former is called their arithmetical, the latter the geometrical relation.[1] It is under this latter view that the ratio and proportion of numbers and magnitudes are considered.

Ratio is defined to be the relation which exists between two magnitudes of the same kind, or two numbers, with respect to quotity. Thus, two lines, two areas, two volumes, two weights, or two abstract numbers, can have a ratio to each other; and the comparison is made by considering what multiple part or parts the first is of the second.

Hence the ratio of two magnitudes of the same kind is represented by the quotient which arises from dividing the units of the first magnitude, called the antecedent, by the units of the second magnitude, called the consequent of the ratio: as, for instances, the ratio of a guinea to a crown, or 21s. to 5s., is $\frac{25}{4}$ or $6\frac{1}{4}$, an abstract number which denotes the number of times, whole or fractional, the antecedent contains the consequent.

It is also obvious that the ratio of any two concrete numbers of the same kind is the same as the ratio of two abstract numbers, as the quotients in both cases are equal.

When the antecedent of a ratio is equal to the consequent, the ratio is called a ratio of equality: when the antecedent is greater than, or less than, the consequent, the ratio is called a ratio of greater or of less inequality.

2. In the seventh book of Euclid's Elements, the definition of proportion is thus expressed:

Four numbers are proportionals, when the first is the same multiple of the second, or the same part or parts of it as the third is of the fourth:

---

[1] The terms arithmetical ratio and geometrical ratio are arbitrary names which do not define nor explain their meaning. The arithmetical relation of two numbers is properly a difference, and may be a concrete or abstract number, of the same nature as the two given numbers. The geometrical ratio is a quotient, always an abstract number. By the ratio of two magnitudes in the latter sense, is meant their *relative magnitude*, how often one contains the other; not their *absolute magnitude*, how much one exceeds the other. Thus, although the absolute magnitude of 12 miles and 1 mile is much greater than that of 12 inches and 1 inch, yet the relative magnitude, or the ratio of the two former, is exactly the same as that of the latter two; or in other words, 1 mile is as small a space in comparison of 12 miles as 1 inch is in comparison of 1 foot.

Or in other words, if the quotient arising from the first of four numbers divided by the second, be equal to the quotient of the third divided by the fourth, the four numbers constitute a proportion.

The definition of proportion has been more briefly defined to be the equality of two ratios.

Thus the four numbers 12, 4, 18, 6 fulfil the conditions required, that the first, 12, has to 4, the second, the same ratio as 18, the third, has to 6, the fourth, and are proportionals;

so that 12 bears the same relation to 4 as 18 does to 6,
which is usually written thus: $12 : 4 :: 18 : 6$.

This same relation holds good when the terms of each ratio are concrete numbers, as 12 yards bear the same relation to 4 yards, as 18 shillings bear to 6 shillings.

The four terms of any proportion may have their order reversed without affecting the proportionality of the numbers, as the fourth, third, second, and first terms of a proportion may be made the first, second, third, and fourth times terms.

Conversely. *If four numbers be proportionals, the quotient of the first divided by the second is equal to that of the third divided by the fourth.*

Thus if 12, 4, 18, 6 form a proportion,
which is usually expressed, $12 : 4 :: 18 : 6$,
Then the ratio $12 : 4$ is $\frac{12}{4}$ or 3,
and the ratio $18 : 6$ is $\frac{18}{6}$ or 3,
whence follows the equality, $\frac{12}{4} = \frac{18}{6}$.

It also appears from the equality of the two fractions, that if the first term of a proportion be greater than the second, the third is greater than the fourth; if equal, equal; and if less, less.

If four numbers, as 12, 4, 18, 6, *be proportionals, the product of the extremes is equal to the product of the means.*

Since 12, 4, 18, 6 are proportionals,
the ratios $\frac{12}{4}$ and $\frac{18}{6}$ are equal, or $\frac{12}{4} = \frac{18}{6}$.

Multiplying these equals by 4×6
Then $12 \times 6 = 18 \times 4$,

or the product of the extremes, 12 and 6, is equal to the product of the means, 18 and 4.

Hence it appears, that if the product of the *means*, 18×4, be divided by 12, one of the extremes, the quotient, 6, will be the other extreme; and if the product of the *extremes*, 12×6, be divided by 4, one of the means, the quotient will be the other mean.[1]

---

[1] Some writers assume the letter x to denote the required number in questions of proportion, and deal with it as a number unknown, but to be determined by the conditions of the question. Others reject the use of all general symbols in numerical computations, as such symbols belong rather to Algebra. Whichever may be the correct view of the matter, there is no doubt that such an assumption affords both facility and convenience in arithmetical reasonings, when it is made the first

And, universally, if any three terms of a proportion be given, the fourth can always be found.

3. *If the product of any two numbers be equal to the product of two other numbers, the four numbers can form a proportion.*

For example, the four numbers 3, 5, 12, 20, are so related that the product of two of them is equal to the product of the other two,

$$\text{Or } 3 \times 20 = 5 \times 12.$$

Dividing these equals by 5 × 20,

$$\tfrac{3}{5} = \tfrac{12}{20},$$

$$\text{And } 3 : 5 :: 12 : 20.$$

It may be noted that when the terms of one ratio are prime to each other, as 3 and 5, they are the least numbers in the proportion.

Since the two equal ratios which constitute a proportion are denoted by two fractions, all the properties of ratios and proportions depend on the properties of fractions.

If the four terms of a proportion be abstract numbers, or concrete numbers of the same kind, the extremes and means may be interchanged one with another, so that the four numbers shall still continue to form a proportion.

Thus the first term shall have the same ratio to the third as the second to the fourth; and the second shall have the same ratio to the first as the fourth to the third.

If, however, the terms of one ratio be either abstract numbers, or concrete numbers of a different kind from the other ratio, this interchange is not possible.

The following properties of a proportion can be readily shewn to be true by means of fractions. Sec. IX.

1. If any equimultiples, or equisubmultiples, of the first and second terms of a proportion be taken, and the same or any other equimultiples or equisubmultiples of the third and fourth be taken, the four numbers will form a proportion.

---

term of the first ratio in questions of proportion. As for example—If 10 pounds of tea cost 25 shillings, what is the cost of 100 pounds?

Here 10 pounds cost 25 shillings.
Let 100 ,, ,, x

Here the number x bears the same ratio to 25 as 100 bears to 10.

$$\text{And } \frac{x}{25} = \frac{100}{10}\text{; and } x = \frac{100 \times 25}{10} = £12\ 10\text{s}.$$

But when the proportion is *inverse*, it appears more convenient to adopt the form in Art. 1 p. 2, as in the following example:—

If 5 men perform a work in 12 days, in how many days can 4 men perform an equal work?

Here the *effects* are the same, and the effective causes must be equal.

If x denote the unknown number of men,

The first effective cause is 5×12
,, second ,, ,, $4 \times x$

$$\text{and } 4 \times x = 5 \times 12,\ x = \frac{5 \times 12}{4} = 15 \text{ men.}$$

2. If any equimultiples or equisubmultiples of the first and third terms of a proportion be taken, and the same or any other equimultiples or equisubmultiples of the second and fourth be taken, the four numbers will form a proportion.

3. The sum or difference of the first and second terms is to the first, as the sum or difference of the third and fourth terms is to the third.

4. The sum or difference of the first and second terms has the same ratio to the second, as the sum or difference of the third and fourth has to the fourth.

5. The sum of the first and second terms is to their difference, as the sum of the third and fourth terms is to their difference.

6. If the corresponding terms of two proportions be multiplied together, or divided, the products will form a proportion.

4. A proportion may be considered either as direct or inverse.

A proportion is direct when the same relation exists *directly* between the first and second terms as between the third and fourth; as 3, 6, 5, 10, are in direct proportion; for the ratio of 3 to 6 and of 5 to 10 is $\frac{1}{2}$; it follows that 3 bears the same direct proportion to 6, as 5 bears to 10; or 3 : 6 : : 5 : 10.

A proportion is inverse when the first and second terms are *directly* proportional to the *inverse* of the third and fourth terms; as 2, 6, 9, 3, are in inverse proportion. For the direct ratio of 2 to 6 is $\frac{1}{3}$, the same as the inverse ratio of 9 to 3; it follows that 2 bears to 6 the same *direct* proportion as the *inverse* of 9 bears to the inverse of 3, or that $2 : 6 : : \frac{1}{9} : \frac{1}{3}$.

The nature of direct proportion may be exemplified, if the names of *cause* and *effect* may be given to those numbers in the proportion which depend directly on one another, whatever may be the causes or effects, provided only that they can be expressed by numbers. As, for instance, a sum of money may be considered the *cause*, and what is gained by it the *effect:* as if £100 produce a gain of £5; at the same rate £500 will produce a gain of £25. Here the first cause bears the same ratio to the second cause as the first effect to the second effect.

The nature of inverse proportion may be illustrated in the same manner. In this case, however, the causes bear an inverse ratio to the distances or the times involved in the question, in which the effects are the same; as, for example: If 12 men can perform a certain work in 5 days, 6 men can perform the same in 10 days. Here the units of cause are inversely proportional to the times. Also in the case of the straight lever, when kept in equilibrium on a fulcrum; the two weights are inversely proportional to their distances from the fulcrum.

In questions, however, in which the different causes produce different effects in given times, the time is an important element in the calculation, as 1 man in 3 days can produce the same effect as 3 men in 1 day. In such cases the *effective cause* is the product of the units of agent and the units of time for producing the given effect.

In all questions involving these three elements there will be found given; one cause, its effect and the time in which it was produced; and the requirement will be; to find *what effect* another given cause will produce in another given time; or what cause will be necessary to produce another given effect in another given time; or in what time another given cause will produce another given effect.[1]

5. Proportion is employed in calculating the interest of money, which is a payment agreed upon between the borrower and the lender for the loan of a sum of money, called the Principal. This payment is generally a fixed sum for every £100 during a year, and is called the rate per cent. The sum of the Principal and the Interest together is called the Amount.

Interest is called *simple,* when it is paid at regular periods, as yearly or half-yearly, or at the end of the time for which the Principal was lent. It is obvious that the simple interest of any sum for one year can be found by a proportion when the interest of £100 for one year is given; and the interest for any number of years will be determined by multiplying the interest for one year by the given number of years.

The result, however, is more readily obtained, by dividing by 100, the product of the principal, the rate per cent., and the number of years.

The interest for months, for weeks, or for days, of any sum will also be found by means of a proportion, from the interest for a year, which may be considered as consisting of 12 months, 52 weeks, or 365 days.

If the amount and the principal be given, the interest is the difference between them. Also of the principal, amount, rate per cent., and time; if any three be given, the fourth can be found by means of a proportion.

Interest is called *compound* when the interest of the principal for the first year is not paid at the end of the year (or of the period) when it is due, but is added to the principal, so that the first year's amount becomes the principal for the second year; and so on for the third and the succeeding years.[1]

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[1] The following question may be taken as an example:—

If 120 men in 15 days of 12 hours long can dig a trench 50 yards in length, 2 in breadth, and 4 deep; how many men will be required to dig a trench 72 yards long, 6 broad, and 5 deep, in 9 days of 10 hours long?

Here, the first cause, 120 men;
effect, $50 \times 2 \times 4$ cubic yards;
time, 15×12 hours.
The second cause, Men required;
effect, $72 \times 6 \times 5$ cubic yards;
time, 9×10 hours.

Taking the effects proportional to the effective causes, the number of men required will be found to be 324.

The requirement may be varied to find the days, the hours, or the effects.

The compound interest of any sum for any number of years may be calculated by proportion, for each successive year, as simple interest.

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[1] In finding the compound interest for any part or parts of a year after the expiration of several years, it is usual, for instance, if the compound interest be required for $4\frac{3}{4}$ years, to take three-fourths of the 5th year's interest. This may be objected to as implying that the interest is payable quarterly. This approximation, however, does not differ much from the exact truth, as will be seen in the following example.

To find the compound interest of £300 for $4\frac{3}{4}$ years at 5 per cent. per annum.

£
$\frac{5}{100}=\frac{1}{20}$ | 300 = First year's Principal.
15 ,, Interest.
$\frac{1}{20}$ | 315 ,, Amount, or 2nd year's Principal.
15·75 Second year's Interest.
$\frac{1}{20}$ | 330·75 ,, Amount.
16·5375 Third year's Interest.
$\frac{1}{20}$ | 347·2875 = ,, Amount.
17·364375 Fourth year's Interest.
$\frac{1}{20}$ | 364·651875 = ,, Amount.
18·23259375 = Fifth year's Interest.
13·6744453125 = $\frac{3}{4}$ of 5th year's Interest.
364·651875 = Fourth year's Amount.
£378·3263203125 = $4\frac{3}{4}$ year's Amount.
20
s. 6·5264062500
12
d. 6·31687500
4
q. 1·267500

£	s.	d.	
378	6	$6\frac{1}{4}$	= Amount.
300	0	0	= Principal.
£78	6	$6\frac{1}{4}$	= Compound Interest for $4\frac{3}{4}$ years.

It may be here remarked that the number of decimal places in this computation might be restricted to 6 or 7, without incurring the error of a farthing.

If the compound interest of £300 for $4\frac{3}{4}$ years, at 5 per cent. per annum, be calculated from the formula $M=P(1+r)^n$, by means of a table of logarithms calculated to seven places of decimals, it appears from the calculation that the correct result is,

£	s.	d.	
378	4	$10\frac{1}{5}$	= Amount.
300	0	0	= Principal.
£78	4	$10\frac{1}{5}$	= Compound interest for $4\frac{3}{4}$ years.

And the approximation is too great by 1s. 8d. nearly.

The compound interest of any sum can be found by a shorter process, by simply taking the same aliquot parts of the principal of each successive year, as the rate per cent. is of £100.

6. Discount is defined to be a deduction made for the present payment of a sum of money due at some future time, and differs from simple interest or Banker's discount.

If £100 be lent for one at 5 per cent. per annum, the amount at the end of the year would be £105.

But if £100 be due at the end of a year at the rate of 5 per cent. interest per annum, and if £5 be deducted for present payment, the sum paid to the creditor would be £95.

Now if this £95 were put out immediately to interest for a year, at 5 per cent.; at the end of the year the amount would be £99. 15s.; whereas if the payment had been deferred till the end of the year, the creditor would have received £100, and would not have lost 5s.; in the transaction he should have lost nothing.

It is therefore clear that £5 was too large a sum to be deducted for present payment. The interest should have been calculated on *the present worth of* £100, *and not on the worth of* £100 *at the end of the year*.

And since the present worth of £105 due at the end of a year is £100, the present worth of £100 due at the end of a year will be a fourth proportional to £105, £100, £100, which will be found to be £95. 4s. $9\frac{1}{7}$d., and the difference £4 15s. $2\frac{1}{7}$d., between this present worth of £100, and £100 due at the end of the year, is the interest of the present worth for the same time and at the same rate of interest. The interest of the present worth is also a fourth proportional to £105, £100, £5; which will be found to be £4. 15s. $2\frac{6}{7}$d.

By means of these principles, may be found the time at which one payment may be made equal to the sum of several payments made at different times, at the same rate, or at different rates of interest. If the present worth of the different sums be found separately for the times and at the rates of interest given, then the time may be found when the sum of the present worths shall be equitably equivalent to the amount of the given sums at the given rate of interest.

It may be added that the present worth and the discount of any sum due any number of years hence, may also be calculated by compound interest by means of a proportion.

7. The purchase and sale of stocks, whether they consist of the national debts of governments, or the nominal capital of trading companies, are all subject to the rule of proportion, as well as all questions which arise on the dividends of such stocks. The national debts of foreign governments in general consist of bonds of £100, or some multiple or submultiple of that sum. They are contracted for a fixed rate of annual interest payable half-yearly, and are so arranged as to be gradually paid off within a certain number of years. They are trans-

ferable from one person to another until they are drawn and paid off. The value of these bonds may be worth more or less than £100 sterling. When a bond of £100 stock is worth £100 sterling, it is said to be at par; and when it is worth less or more than £100 sterling, it is said to be below or above par.

The national debt of the United Kingdom consists of stocks called consols (consolidated annuities), of which the chief stock is called the 3 per cent. consols, which bear an annual interest of £3 sterling, payable half-yearly, for every £100 stock. This stock has no period fixed for its redemption like foreign stocks, but may be purchased and sold at any time at the ordinary rates of the market.

The exchange of a sum of money of one nation into the equivalent to that of another, is effected by a proportion when the equivalent values of the units of the money of the two nations are given; and when several exchanges are made in one transaction, these separately constitute a repetition of the same process.

The computations in all mercantile transactions depend upon the principle of proportion. And the profit or loss on all sales and purchases are in general computed at some fixed rate for every £100, as are also all charges for commission, brokerage, &c., for the purchase or sale of merchandise. The equitable division of the profits or losses of a trading speculation can be effected by means of proportion; and in cases where the element of time enters into the transaction, both the time and the capital employed must be considered as constituting the efficient cause of the effect produced.

In the same manner the assets of a bankrupt can be divided among his creditors according to the proportion of their claims on the bankrupt's estate.

EXERCISES.

I.

1. Explain what is meant by ratio, and state under what conditions two concrete numbers can have a ratio to one another.

2. What ratio does 63 acres, 2 roods, 19 poles, $22\frac{1}{4}$ yards, bear to 199 acres, 3 roods, 33 poles, $13\frac{3}{4}$ yards?

3. When are four magnitudes said to be proportionals? Apply the definition as a test to ascertain whether the four quantities 3lb. 2oz., 1s. $1\frac{3}{4}$d., 1s. $7\frac{1}{2}$d., 4lb. 2oz., can be so arranged as to form a proportion.

4. What number has the same proportion to 360 as 231 has to 72?

5. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.

6. Determine a fourth proportional to 3·125, ·000125, and ·0732.

7. Shew that 6yds. 3qrs. bears the same proportion to 73yds. 2qrs. as 5s. 3d. does to £2. 17s. 2d.

8. Find a sum of money which shall be the same part of £61. 9s. 1d., that 2cwt. 2qrs. 10lb. is of 36cwt. 1qr.

9. Increase £5. 4s. 6d. in the proportion of 11 to 19, and state how much per cent. is added.

II.

1. Find the cost of 4tons 2cwt. 1qr. 11lb. of merchandise at £3. 17s. per hundredweight.

2. If 5cwt. 3qrs. 14lb. cost £6, what will be the cost of one pound?

3. What is the value of 39cwt. 0qr. 10lb. at £1. 17s. 10d. per cwt.?

4. What is the cost of 17cwt. 1qr. 12lb. at 9s. 11d. per quarter?

5. If $\frac{3}{4}$ of a yard cost 9s. 4d., what number of yards will £1·00 purchase?

6. What is the rent of 247 acres of land for one year and 146 days at £1. 13s. 6d. per acre?

7. A merchant bought 4 pipes of wine and sold one pipe for £50, but by doing so he lost 5 per cent.; at what rate must he sell the remaining 3 so as to gain 20 per cent. by the whole?

8. A man pays a corn rent of 5qrs. of wheat and 3 of barley Winchester measure; what is the value of his rent, wheat being at 60s., and barley at 54s. the quarter, Imperial measure? supposing 33 Winchester gallons equivalent to 32 Imperial ones.

9. A tenant holds a farm of 350 acres subject to a tax of 3s. 6d. per acre yearly, and a corn rent of 100 quarters of wheat, barley, oats, and beans respectively. Find the amount of his rent when the

average prices of wheat, barley, oats, and beans per quarter are 48s. 9d., 30s. 4d., 23s. 6d., and 33s. 5d. respectively.

10. If the sixpenny loaf weigh 4·35lb. when wheat is at 5·75s. per bushel, what ought to be paid for 49·31lb. of bread when wheat is at 18·4s. per bushel?

11. If 5 pipes of wine be bought for £300, and one pipe be damaged, at how much per gallon must the remainder be sold so as neither to gain or lose by the transaction?

III.

1. What is the simple interest and amount of £305 in 6 years at 4½ per cent. per annum?

2. In what time will £305 amount to £387. 7s. at 4½ per cent. per annum simple interest?

3. If £305 amount at simple interest in 6 years to £387. 7s., what is the rate per cent. per annum?

4. What principal put out to simple interest at 4½ per cent. for 6 years will amount to £387. 7s.?

IV.

1. Required the amount of £56. 13s. 4d. put out to simple interest for 5 years and 4 months at 6 per cent. per annum.

2. What is the commission on £529. 18s. 5d. at 2¼ per cent.?

3. The amount of £500 in ¾ of a year was £520. Required the rate per cent.

4. What is the interest of £275. 10s. for 219 days at 5 per cent. per annum?

5. If money bear interest at 5 per cent. per annum, of what sum is one groat the daily interest?

6. In the year 1248 A.D., the money-lenders were not allowed to charge the scholars at Oxford a higher interest for the loan of money than twopence for the use of one pound for a week; how much was that per cent. per annum?

7. An estate is bought at 25 years' purchase for £15,000, two-thirds of the purchase-money remaining on mortgage at 3 per cent.; the cost of repairs averages £100 a year. What interest does the purchasor make on his investment?

8. If £5 be paid for the loan of £100 for one year, what ought to be paid for the loan of £275 for 5 years?

9. If A lend B £150 for 7 months, and when B has occasion to borrow from A, the latter can spare only £70; how long may B keep the £70 until the interest is equal to the interest of A's loan?

10. Which is the greater rate of interest, £7 for the use of £145 or £4½ for the use of £91 5s. for a year?

11. The simple interest of £25 for 3½ years was found to be £3. 18s. 9d.; required the rate per cent. per annum.

12. If £960 put out for 6 years at simple interest amount to £1248; what is the rate per cent. per annum?

13. In how many years will £800 become £1088 at 4 per cent. simple interest?

V.

1. What sum of money is that which amounts in 3 months to £249. 9s., and in 9 months to £256. 3s. 8d., and at what rate of interest is it lent?

2. What was the interest of the national debt of Great Britain and Ireland for one hour, calculated at the rate of 3 per cent. per annum, if the amount in 1872 was £730,986,800?

3. What must be the rate of interest per cent. per annum, in order that the interest of fifty American dollars may be one cent. per diem?

4. What is the interest on 30·029 rupees, 4 annas, 6 pice, at $4\frac{1}{2}$ per cent.? Give the result in English money when the rupee is worth 2s. $1\frac{1}{2}$d., the rupee containing 16 annas, and the anna 12 pice.

VI.

1. Explain the distinction between true discount and banker's discount. Does the creditor or the debtor gain by computing interest instead of discount?

2. Find the discount on £100 due one year hence, if money bear interest at 5 per cent. per annum. Calculate the interest on this discount for the same time, and shew that it is equal to the difference between the interest and the discount of £100.

3. Find the present worth of £250 due $2\frac{1}{2}$ years hence at 5 per cent. per annum; and shew that the discount of the given sum is equal to the present worth for the same time and at the same rate of interest.

4. A person agrees to discount a bill for £675 payable in 40 days at the rate of $2\frac{3}{8}$ per cent. per annum interest. How much does he pay?

5. What must be the rate of interest, in order that the discount on £2573 payable at the end of one year and seventy-three days may be £93?

6. Shew that the interest on £266. 13s. 4d. for three months, at $4\frac{1}{2}$ per cent. per annum, is equal to the discount of £83 for 15 months at 3 per cent. per annum.

7. If the discount on a bill due 8 months hence at $2\frac{1}{2}$ per cent. per annum be £12. 0s. 11d., what is the amount of the bill?

8. What sum of money must be placed out at simple interest for 5 years at $4\frac{1}{2}$ per cent. that the amount may be £1000?

9. What is the present worth of £1275, payable as follows: £90 in 3 months, £735 in 7 months, and the rest in 11 months, if money bear interest at 5 per cent. per annum?

10. A man having lent £2,500 at 5 per cent. interest, payable half-yearly, wishes to receive his interest in equal portions monthly, and in advance; how much ought he to receive every month?

11. If the principal sum with interest at the rate of 5 in the 100 for a month amount in a year to 1000; tell the principal and interest respectively (*Lilavati*).

12. Find the discount on £170. 18s. 5d. due 52 days hence, at $2\frac{1}{2}$d. per £100 per day.

13. If £10 be the interest of £110 for a given time, what should be the discount of £110 for the same time?

14. Shew that the interest on the discount of £100 for one year at 5 per cent. per annum is equal to the discount on the interest at the same rate for the same time.

VII.

1. Find the amount of £320. 10s. for 4 years, at 5 per cent. compound interest, when the interest is payable yearly and half-yearly.

2. Find the amount of £819. 4s. in 6 years at £12. 10s. per cent. per annum compound interest.

3. Find the amount of £62. 10s. in four years at 20 per cent. per annum compound interest.

4. What sum must be paid down to receive £600 at the end of three years, allowing 5 per cent. per annum compound interest?

5. Find the compound interest of £250 at 3 per cent. per annum for 2 years and 195 days.

6. What is the compound interest of £410 for the term of $2\frac{1}{2}$ years at $4\frac{1}{2}$ per cent. per annum, the interest payable quarterly?

7. The difference between the compound and simple interest of a certain sum of money for 3 years at 4 per cent. is 19s. Find the sum.

8. Find at what rate of simple interest in two years a sum would amount to the same sum as at 4 per cent. compound interest.

9. Compare the simple and compound interest of £350 for 4 years at $4\frac{3}{8}$ per cent. per annum.

10. In how many years will £1000 amount to £1123. 12s. at 6 per cent. per annum compound interest?

11. A person put out to interest £2,000 at 4 per cent.; he spends annually £75, and adds the remainder of his dividend to his stock. What is he worth at the end of 5 years?

12. What sum of money put out to compound interest for 5 years at 5 per cent. per annum will amount to £100?

13. What is the present value of the compound interest, 3 per cent. per annum, of £100, to be received five years hence?

14. Find the whole amount and the interest accumulated at the end of 4 years, by one who invests £200 at the beginning of each year at $3\frac{3}{4}$ compound interest.

15. If a legacy of £500 be left to a youth of 15 years of age, and invested at 4 per cent. per annum compound interest; what sum would he receive when he becomes of age?

16. A person saves from his income £100 a year, and invests his

savings at 5 per cent. per annum compound interest; what amount would be accumulated in ten years?

17. An estate was mortgaged for £2079. 15s., upon which interest was payable at $4\frac{3}{4}$ per cent. per annum. Supposing no payments made until the expiration of $7\frac{1}{2}$ years, when the whole debt was cleared off, find the accumulated amount due at that time.

18. If a sum of money were put out to interest at 5 per cent. per annum, in how many years would it double itself (1) at simple interest, (2) at compound interest?

19. A sum of money was put out at 5 per cent. per annum compound interest; and the interest in four years was £107 .15s. $0\frac{3}{4}$d. What was the sum?

VIII.

1. What is meant when it is said that consols are at $88\frac{2}{8}$? What are they at when £9000 is paid for £10000 consols?

2. What sum will be required to purchase £250 stock in the 3 per cent. consols when they are at $95\frac{3}{8}$ per cent.? And conversely, what stock in the 3 per cent. consols can be purchased for £238. 8s. 9d., when they are at $95\frac{3}{8}$? What would be the difference in each case by taking into the account the commission of the broker at $\frac{1}{8}$ per cent.?

3. If the 3 per cent. consols are at 94, what should be the price of the 4 per cents to give the same rate of interest?

4. If the 3 per cents be at 96, and the $3\frac{1}{2}$ per cents at 103, which stock gives the greater interest?

5. A person invests £4600 in the 3 per cents at $91\frac{7}{8}$, and sells out at $95\frac{1}{8}$ a year afterwards. What percentage does he get for his money, the broker charging $\frac{1}{8}$ per cent. for both the purchase and sale?

6. Which is the better investment, bank stock paying 10 per cent. at 317, or 3 per cent. consols at 95?

7. A person holds as much stock of a certain kind as is worth £879, the price being £97. 10s. for £100 stock; there is also another stock which sells at £88. 5s. for £100; find how much of the latter stock he ought to receive in exchange for his property in the former.

8. Which is the more profitable investment, the purchase at £96 of 3 per cent. consols, or the purchase of shares in an insurance office at £227 per share, the annual dividend on a share being £7. 10s.?

9. Shew that the interest obtained by investing a sum of money in the 3 per cents at $82\frac{1}{2}$, is to the interest obtained by investing the same sum in the $3\frac{1}{2}$ per cents at $93\frac{1}{2}$, as 34 to 35.

10. What amount of stock in the $3\frac{1}{2}$ per cents will give the same annual income as £3560. 8s. of the 3 per cent. stock?

11. What must be the price of stock in the 3 per cent. consols, that capital invested therein may produce 5 per cent. per annum?

12. If £1000 be invested in the purchase of 3 per cent. stock at $81\frac{5}{8}$, at what rate must the stock be sold to gain £100?

13. A person by selling out of the 3 per cents at 99 realises £15,345, and gains 10 per cent. on his investment by the transaction. At what price did he buy?

14. If £1000 be invested in the 3 per cent. consols when they were at 85, and some time after the stock was sold out, and it was found that £100 was lost in the sale; what was the value of £100 stock?

15. A person buys £750 stock in the 3 per cents when they are at $98\frac{3}{4}$, and 6 months after sells it at 103; what did he gain?

16. *A* having £2000, purchased (May 22) stock in the 3 per cents, then at $96\frac{3}{8}$, but sold out again shortly afterwards when they had fallen to 94. What does *A* lose; and what is the portion of the half-yearly dividend paid in July 6 that accrues after *A's* purchase?

17. How much stock must be bought at $85\frac{1}{2}$ per cent., in order that by selling it out when stocks are at $90\frac{3}{4}$, twenty guineas may be gained?

18. The three per cent. consols were at $98\frac{1}{8}$, and £740 of this stock, including commission, cost a person £717. 8s.; how much per cent. did he pay for commission?

IX.

1. A person sells out of the 3 per cents at 96, and invests the money in railway 5 per cent. stock. By this means his income is increased 15 per cent. What is the price of the railway stock?

2. A person sells out of the 3 per cent. consols at $91\frac{1}{8}$, £700 stock. After consols have fallen $2\frac{1}{2}$ per cent., he buys in again with the same money. What difference will thus be made in his annual income?

3. *A* invests £11,025 in the purchase of 3 per cent. stock at $87\frac{1}{2}$, and *B* invests the same sum in the purchase of 4 per cent. stock at $94\frac{1}{2}$; what will be the difference of the net incomes after deducting income tax at 10 pence in the pound?

4. How much 4 per cent. stock can be purchased by the transfer of £1000 stock from the 3 per cents at 72, to the 4 per cents at 90?

5. When the 3 per cents are at $91\frac{5}{8}$, and the $3\frac{1}{2}$ per cents at $99\frac{1}{4}$, which will be the better investment? How much is one investing when the difference in income is a shilling?

6. A person transfers £1000 stock from the 5 per cents to the 3 per cents when the former is at 110 and the latter at 84; if, at the end of six months, the 5 per cents have risen to 112, what must then be the price of the 3 per cents, that he may sell out without having gained or lost by the transfer?

7. A person sells £1200 stock in the 3 per cents. at £86, in order to invest in bank stock paying 8 per cent.; what price must he pay for it to be neither a gainer nor loser?

8. Would a person increase or diminish his income by selling £1157 3 per cent. stock at $83\frac{1}{4}$ to purchase into the $3\frac{1}{4}$ per cents at $83\frac{7}{8}$?

9. A person having his property in the Great Eastern Railway £100 stock, which is at $52\frac{1}{8}$, and pays a half-yearly dividend of $1\frac{5}{8}$ per

cent. He sells out and invests in the 3 per cents at $95\frac{7}{8}$; will he increase or diminish his income?

10. A person sells out £153,000 stock, 3 per cent. consols at $92\frac{1}{4}$, and invests the proceeds in a foreign 5 per cent. stock at $102\frac{1}{2}$; when each stock has fallen $5\frac{1}{8}$, he again sells out, and reinvests in the 3 per cents; by how much will the annual income differ from what it was at first?

X.

1. £400 is due at the end of two years, and £2100 at the end of eight years; what is the equated time for one payment, reckoning 5 per cent. simple interest?

2. If £40 be due at the end of 6 months, £60 at the end of 12 months, and £80 at the end of 15 months; what is the equated time for paying the whole, allowing interest at 3 per cent. per annum?

3. Supposing £100 due now, and £1000 at the end of 10 years; required the equated time of payment at 5 per cent. per annum.

4. *A* is under an engagement to pay *B* the sum of £400 at the end of 2 years, and of £2,100 at the end of 8 years. At what time may he pay both sums together without loss or gain, reckoning at 6 per cent. simple interest?

XI.

1. Two persons contribute £8000 and £12,000 respectively to a joint trading stock; the gain is £5000: what is the share of each?

2. A bankrupt's estate is £500; he has three creditors whose claims are £154. 10s., £320. 5s., and £415. 5s. respectively; divide it among them.

3. Three persons contribute £1000, £1200, £1780 respectively, and after trading 14 years, at the dissolution of the partnership the firm was found worth £180,000; what did each man receive?

4. The estate of a bankrupt worth £21,000 is to be divided among four creditors; the debts due to *A* and *B* are as 2 : 3, to *B* and *C* as 4 : 5, and to *C* and *D* as 6 : 7; what must each receive?

5. *A* puts into a speculation £500 for 18 months, and *B* £1000 for 12 months; they gained £250: find the share of each.

6. Three merchants traded in partnership; one put into the concern £1500 for $3\frac{1}{2}$ years, another £1200 for 3 years, and the third £1000 for 21 months, and they gained £1000; what was the gain of each partner?

7. *A*, *B*, *C* enter into business with capitals of £625, £925, and £1200 respectively, and at the end of a year divide profits to the amount of £687. 10s.; what is the share of each?

8. *A* and *B* were partners for 12 months. *A* advanced £400 for the first 3 months, and then £750 more; *B* advanced £500 for the first 5 months, and then £450 more. They gained £1020; what should each receive?

9. *A* and *B* rent a field for £60. *A* puts in 10 horses for $1\frac{1}{2}$ months,

30 oxen for 2 months, and 100 sheep for $3\frac{1}{4}$ months; *B* puts in 20 horses for 1 month, 40 oxen for $1\frac{1}{2}$ months, and 200 sheep for 4 months. If the food consumed in the same time by a horse, an ox, and a sheep be in the ratio 3 : 2 : 1; find the portion of the rent of the field which each must pay.

10. Three merchants *A*, *B*, *C*, gain £5000 by trading in 25 months. *A's* capital was £2400, *B's* £3500, and *C's* £4500. *B* put in his capital for the whole time, *A* having put in his for the last 15 months, and *C* for the last 12 months; what should each receive?

11. *A*, *B*, and *C* rent a pasture for a year for 20 guineas: *A* puts in 25 cattle for the whole time, *B* 30 for 9 months, *C* 45 for 7 months; find the rent paid by each.

12. Three persons begin trade, each by investing £500 in the concern. *A* adds £50 more at the end of every two months, *B* adds £75 at the end of every three months, and *C* £100 at the end of every four months. If the profits be £661. 10s. at the end of the year, what is each partner's share?

13. A bankrupt has three creditors, to whom the sums due are as the numbers 3, 4, 5; if his assets are valued at £6000, find the sums they will respectively receive.

14. *A* and *B* contract to execute a certain order for £1245. *A* employs 100 children for 3 months, 80 women for 2 months, and 40 men for 1 month; *B* employs 120 children for 2 months, 60 women for $1\frac{1}{2}$ months, and 80 men for $2\frac{1}{2}$ months. If the work done in the same time by a child, a woman, and a man be in the ratio 1 : 2 : 3, find the sum of money which *A* and *B* must each receive.

XII.

1. The exchange between London and Paris is $25\frac{1}{2}$ francs for one pound sterling; between Paris and Amsterdam 117 francs for 55 florins; between Amsterdam and Hamburgh 11 florins for 13 marks: what is the exchange between London and Hamburgh?

2. If 3lb. of tea cost as much as 17lb. of sugar, and 10lb. of sugar as much $1\frac{1}{2}$lb. cloves; how many pounds of tea must be given for 100 pounds of cloves?

3. If 6 pounds of pepper be worth 13 pounds of ginger, and 19 pounds of ginger be worth $4\frac{1}{2}$ pounds of cloves, and 10 pounds of cloves be equivalent to 63 pounds of sugar at 5d. per pound; what is the value of one hundredweight of pepper?

4. If 7 oxen are worth 45 sheep, and 3 sheep cost 10 guineas, what will 8 oxen cost?

5. If 7 oxen be worth 42 sheep, and 3 sheep be worth 16 turkeys or 40 geese; how many oxen must be given in exchange for 290 turkeys and 715 geese?

XIII.

1. Two liquids cost 7 and 9 shillings a gallon respectively. At

how much per gallon must a mixture of two-thirds of the former and one-third of the latter be sold for the dealer to gain 20 per cent.?

2. A tea-dealer has teas worth 4s. 6d. and 3s. 6d. per pound respectively, which he mixes, taking two pounds of the former to one pound of the latter, and sells the mixture at 4s. 4d. per pound; what does he gain or lose per cwt.?

3. If 1000 pounds of sugar at 8d. a pound, 1100 at 7d., and 1500 at 6d. be mixed together: at what rate per pound should the mixture be sold to gain 20 per cent. on the outlay?

4. A grocer mixes 9lb. of coffee at 2s. 3d. a pound with 6lb. of chicory at 7½d. a pound; at what price must he sell the mixture to gain 25 per cent.?

5. A tea-dealer mixes together 3lb. of tea at 5s., 4lb. at 6s., and 5lb. at 7s., and sells the mixture at 6s. 9¼d. per pound; how much per cent. does he gain by the transaction?

XIV.

1. How much per cent. is 7½d. in the pound?
2. If gain be 2½d. in the shilling, what is that per cent.?
3. How much per cent. is £8. 6s. 8d. in £26. 13s. 4d.?
4. A person paid a tax of 10 per cent. upon his income. What must his income have been if, after paying the tax, he had £1250 left?
5. What is the clear value of a legacy of £2000 in the 3 per cents, when they are at 92⅓, the bequest being subject to a duty of 10 per cent.?
6. A legacy of £1000 is left to 3 individuals in the proportion of 1, 2, 3; find the sums received by each after deducting the legacy duty of 10 per cent.
7. If an estate be worth £1500 a year and the land tax be assessed at 2s. 8½d. in the pound; what is the clear income?

XV.

1. If sugar be bought at 9d. per pound; at what rate must it be sold to gain 25 per cent.?
2. If cloth be bought at 7s. 6d. a yard; at what rates must it be sold (1) to gain, (2) to lose 15 per cent.?
3. If an article which costs a tradesman £18 be sold after 4 months for £25; what was the gain per cent. per annum?
4. If 208 yards of cloth be bought at 12s. 6d. a yard, and 80 yards be sold at 15s., and 98 at 14s.; at what price must the remainder be sold to gain 12 per cent. on the outlay?
5. By selling an article which cost £14 per cwt. at 2s. 9¾d. per pound, 5 per cent. more profit is gained than if the whole was sold for £55. 15s. 3¾d. What was the amount sold?
6. By selling tea at 5s. 4d. per pound, a grocer clears one-eighth of

his outlay, he then raises the price to 6s.; what does he clear per cent. by the latter price?

7. If 14 per cent. be lost in the sale of goods for 8s.; what was the loss per cent. if sold for 7s. 6d.?

8. If 5 per cent. be lost in selling cloth at 10s. a yard; how much is gained or lost per cent. by selling it at 12s. 6d. a yard?

9. In a sale of goods for £182 there is a loss of 9 per cent.; for what must 3 times the quantity be sold in order to gain 7 per cent.?

10. If 20 per cent. be gained by selling an article for 10s. 6d.; what is the gain or loss per cent. when it is sold for 8s.?

11. When goods are sold for £80, the gain per cent. is 10; what should be the price of an equal quantity of the same article that the gain per cent. might be 19?

12. Goods are bought for £142. 10s., and sold for £163. 13s. 11$\frac{1}{4}$d.; what is the gain per cent.?

13. A grocer had 150lb. of tea, of which he sold 50lb. at 9s. per pound, and found he was gaining only 7$\frac{1}{2}$ per cent. But he wished to gain 10 per cent. on the whole. At what rate must the remaining 100lb. be sold that he may attain his wishes?

XVI.

1. A tradesman adds 35 per cent. to the cost price of his goods, and gives his customers a reduction of 10 per cent. on their bills; what profit does he make?

2. A merchant bought goods for £170, and sold them again for £210 to be paid at the end of nine months. What was his gain in ready money discounting at 6 per cent.?

3. A tradesman finds that if he asks for his goods 15 per cent. above the wholesale price, he can sell his whole stock in 4 months, whereas if he asks 20 per cent. he requires 6 months to sell the same amount. Which will he find the more profitable system at the year's end?

4. A bill of £630 due a year hence can be taken up now at 5 per cent. discount. Supposing that a tradesman can employ his capital so as to obtain interest at the end of every quarter at the rate of 4$\frac{1}{2}$ per cent. per annum, had he better so employ it, or take up the bill? and what will be the difference to him?

5. A tradesman marks his goods with two prices, one for ready money, and the other for one year's credit, allowing discount at 5 per cent. If the credit price be marked 12s. 3d., what ought to be the cash price?

6. If a manufacturer sell an article of which the first cost is £100 to a wholesale dealer at 10 per cent. profit, the wholesale dealer to the retailer at 15 per cent. profit, and the retailer to the consumer at 30 per cent. profit; what sum is paid by the consumer as profits in addition to the first cost of the article?

XVII.

1. If 5 per cent. be lost by selling a horse for £38, at what price must 3 others, which cost each the same as the first, be sold in order to gain 10 per cent. on the whole?

2. Paper money is at a discount of £20 per cent. A man buys goods marked £6. 10s. (paper money) and tenders that sum in gold. How much paper money must he receive in change, 5 per cent. being allowed for present payment?

3. A labourer saves $2\frac{1}{2}$ per cent. weekly out of his earnings, which, in money, were 15s. a week. In consequence of a strike his wages were raised 2s. a week, but he found his cost of living, with his weekly payment to the trades' union, were increased in the ratio of 7 to 6: find how much his yearly savings were increased after he had joined the trades' union.

4. How much stock must be sold out of the 3 per cents at $92\frac{2}{3}$ to pay a bill of £515 nine months before it becomes due, discount being allowed at the rate of 4 per cent. per annum?

5. If a cask of 142 gallons of wine purchased ten years ago for £70 was found to have lost a quart each year by leakage and evaporation; at what price per gallon should the wine be sold to clear 10 per cent., allowing interest on the purchase-money at the rate of 5 per cent. per annum?

6. A horse bought for £88 sells for £110; what is the gain per cent.? If bought for £110, he had been sold for £88, what would have been the loss per cent.? If the rates of loss and gain differ, explain why they differ.

7. Bought a quantity of goods for £150 ready money and sold them again for £200 payable $\frac{3}{4}$ of a year hence; what was the gain in ready money, allowing discount at $4\frac{1}{2}$ per cent.?

8. If goods be sold on the condition to allow 10 per cent. discount, if payment be made at the end of 6 months; what discount ought to be allowed if payment be actually made (1) three months *before*, and (2) three months *after* the stated time, if money bear interest at 5 per cent. per annum?

9. A person purchases goods at 6s. per pound Troy weight and sells them again by Avoirdupois weight; at what rate per ounce must he sell so as exactly to reimburse his outlay?

XVIII.

1. If an ounce of gold be worth £3. 17s. $10\frac{1}{2}$d., and if silver be worth 5s. 2d.; what is the relative value of gold and silver?

2. What is the value of a talent of silver, if silver be worth 5s. per oz., and a talent consist of 1000 shekels, each weighing 219 grains?

3. A wedge of gold is worth £150 at the rate of £3. 17s. $10\frac{1}{2}$d.

per ounce; what is the worth of a piece of silver of the same weight at £2. 14s. 6d. per pound?

4. If gold be at a premium of 20 per cent., and a person buys goods marked 135 dollars, and offers gold to the amount of 135 dollars, what change ought he to receive in notes, 5 per cent. being abated for ready payment?

5. A gold coinage, of which each piece weighs 132 grains, is alloyed with a metal of one-twelfth the value of gold; and another, each piece of 104 grains alloyed with a metal one tenth the value of gold; find the relative value of the two pieces, if a thirteenth part of each coin be alloy.

6. Find the proportion of the values of a gold coin and a silver coin, if 13 gold coins together with 12 silver coins are worth 3 times as much as 3 gold coins and 40 silver.

XIX.

1. If £39 are equal to 1000 francs, the French hectolitre equals ·344 quarters. Is corn cheaper in England at 39 shillings a quarter, or in France at 17 francs the hectolitre?

2. If 8000 metres be equal to 5 miles, and if a cubic fathom of water weigh six tons, and a cubic metre of water 1000 kilogrammes; find the ratio of a kilogramme to a pound Avoirdupois.

3. A block of mahogany in the form of a rectangular parallelopiped measures along its edges $18\frac{1}{2}$ feet, $5\frac{1}{4}$ feet, and 3 feet respectively; determine its value on the supposition that a cubical block of the same wood measuring 7 inches along the edge is worth 3s. 6d.

4. A beam is 5ft. 6in. long, 10in. wide and 8in. thick, and weighs 8cwt. 1qr.; find the length of another beam the end of which is 100 square inches which shall weigh one ton.

5. If a piece of land 375ft. 6in. long, and 75ft. 9in. in breadth, cost £118. 2s. 6d.; what will be the price of a piece of similar land 278ft. 9in. long and 157ft. broad?

6. If granite be sold at 2s. $7\frac{3}{4}$d. the cubic yard; find the price of 235 blocks of it, each of which contains $43\frac{1}{2}$ cubic feet.

XX.

1. A piece of work can be done in 50 days by 35 men working all together; and if after working together for 12 days 16 of the men were to leave the work, find the number of days in which the remaining men could finish it.

2. The expense of 19 workmen for 35 weeks is £310. 10s.; how many weeks must 43 workmen labour for £555. 7s.?

3. A person contracts to do a piece of work in 30 days and employs 15 men upon it; the work is half finished in 24 days: how many additional workmen must he then introduce in order to perform the contract?

4. If 5 oxen or 7 horses eat up the grass of an enclosure in 74 days; in what time could 5 horses and 7 oxen eat up the grass of the enclosure?

5. If £17. 10s. be gained by a principal of £75 in 9 months; at what rate is that per cent. per annum?

6. If a tradesman with a capital of £2000 gain £50 in 3 months; what sum will he gain with a capital of £3000 in 7 months?

7. If 15 horses and 148 sheep can be kept 9 days for £75. 15s.; what sum will keep 10 horses and 132 sheep for 8 days, supposing 5 horses to eat as much as 84 sheep?

8. If 5 men can reap a field whose length is 800 feet and breadth 700 feet in $3\frac{1}{2}$ days of 14 hours each; in how many days of 12 hours each can 7 men reap a field 1800 feet long and 960 broad?

9. If 25 labourers can dig a ditch 220 yards long, 3ft. 4in. wide, and 2ft. 6in. deep, in 32 days, when the day is 9 hours long; how many labourers would be able to dig a ditch half a mile long, 2ft. 4in. deep, and 3ft. 6in. wide, in 36 days, when the day is 8 hours long?

XXI.

1. If a person travels over 120 miles in 9 days, when the days are 8 hours long; in how many days of 16 hours each will he travel over 640 miles?

2. A is 20 miles behind B, A travels at the rate of $5\frac{1}{4}$ miles an hour, and B at the rate of $3\frac{1}{2}$ miles an hour; after what time will A overtake B?

3. If 54·32 Irish miles are equivalent to 69·14 English; how many English correspond to $15\frac{1}{2}$ Irish?

4. What fraction of an inch on a globe of 50 inches diameter would represent the altitude of Chimborazo, 21,424 feet high, the highest point of the Andes, supposing the earth a sphere, and its diameter 8000 miles?

5. If the shadow of a tower be 80ft. 6in., and that of a stick 3ft. long placed perpendicularly at the same minute be 5ft. 2in.; find the height of the tower, and state the Prop. in Euclid which you employ.

6. The ratio between the polar and equatorial diameters of the earth is 283·33 to 299·33, and the length of the polar diameter is 7899·114 miles; find the length of the equatorial diameter.

7. Assuming that the circumference of a circle is to its diameter as 22 to 7, and that the circumference of the earth is to its diameter as 160 metres to 167 feet, determine to five places of decimals the ratio of a metre to a foot.

8. How much larger does the full earth appear to the inhabitants of the moon (if there be any) than the full moon to the inhabitants of the earth, supposing the diameters of the earth and moon are 8000 and 2000 miles respectively, and that the areas of circles are proportional to the squares of their diameters?

RESULTS, HINTS, ETC., FOR THE EXERCISES.

I.

1. Art. 1. 2. The ratio is 7 to 22. 3. See Art. 2. 4. 1155. 5. A number is called a third proportional to two numbers, when the second number forms the consequent of the first ratio and the antecedent of the second. The third proportional is $\frac{25}{36}$. 6. ·00002928. 7. The first ratio is represented by $\frac{27}{294}$, the second by $\frac{63}{686}$, each of which is equal to $\frac{9}{98}$. 8. 2cwt. 2qrs. 10lb. $=\frac{290}{4060}=\frac{1}{14}$ of 36cwt. 1qr. 9. £5 4s. 6d. is augmented to £9 0s. 6d., which gives an addition of £3 16s. This addition on £5 4s. 6d. gives an addition of $72\frac{8}{11}$ per cent.

II.

1. 4 tons 2cwt. 1qr. 11lb. = 9223lb. and 112lb. cost 77s. Let 9223lb. cost xs., the prices are directly proportional to the weights, and $\frac{x}{77}=\frac{9223}{112}$; $x=\frac{9223\times 77}{112}$s. Sometimes questions of proportion may more readily be answered by the method of practice, as it is called, by multiplying the price of the given unit by the number of units, and making the smaller units some aliquot part or parts of the given unit, as in this case, 1cwt. is the given unit, 1qr. is $\frac{1}{4}$ of 1cwt., 7lb. is $\frac{1}{4}$ of 1q., and 4lb. is $\frac{1}{7}$ of 1qr. If $\frac{1}{4}$ of the price of 1cwt., $\frac{1}{4}$ of the price of 1qr., and $\frac{1}{7}$ of the price of 1qr. be added to 82 times the price of 1cwt., the same result will be obtained.

2. $2\frac{1}{2}$d. 3. £73 18s. $10\frac{15}{28}$d. 4. £19 19s. $2\frac{4}{7}$d. 5. $160\frac{5}{7}$. 6. £414 14s. $3\frac{1}{2}$d. 7. 10s. $8\frac{1}{2}$d. per gallon, or $£67\frac{31}{51}$ per pipe. 8. £22 8s. 9. £741 5s. 10. If x denote the price of 49·3lb. of bread when wheat is at 18·4s. per bushel; since the price of 1lb. of bread is directly proportional to the price of a bushel of wheat, the proportion $\frac{x}{49{\cdot}3}:\frac{6}{4{\cdot}35}::18{\cdot}4:5{\cdot}75$ will give 18s. $1\frac{1}{2}$d. the price of 43·3lb. of bread. 11. $15\frac{55}{63}$s.

III.

1. The simple interest is £82 7s., and the amount £387 7s.

2. Let £82 7s. or 3294 sixpences = interest of 305 for x years.
£13 14s. 6d. or 549 sixpences = interest of 305 for 1 year.

$$\text{Then } \frac{x}{1}=\frac{3294}{549}, \text{ or } x=6 \text{ years.}$$

3. £13 14s. 6d. or 549 sixpences = int. of 305 for 1 year.
Let x = int. of 100 for 1 year.

$$\frac{x}{549}=\frac{100}{305}=\frac{20}{61}$$

$$x=\frac{20\times 549}{61}=180 \text{ sixpences}=£4\tfrac{1}{2} \text{ per cent.}$$

4. £127 or 2540s. = amount of £100 in 6 years.
Let £387 7s. or 7747s. = amount of x in 6 years.

$$\frac{x}{100}=\frac{7747}{2540};\ x=\frac{7747\times 100}{2540}=£305 \text{ the principal.}$$

IV.

1. Amount £74 16s. 1d. 2. £11 16s. $5\frac{1}{2}$d. 3. $6\frac{2}{3}$ per cent. 4. 219 days is $\frac{219}{365}=\frac{3}{5}$ of 1 year. One year's interest is £13 15s. 6d. 5. £130. 6. £43 6s. 8d. per

cent. per annum. 7. £1 6s. 8d. per cent. 8. £68 15s. 9. 15 months. 10. The second is a little greater than the first. 11. $4\frac{1}{2}$ per cent. per annum. 12. 5 per cent. per annum. 13. In 9 years.

V.

1. The sum is £246 1s. 8d., the rate per cent. is $5\frac{1395}{2953}$. 2. £2501 13s. $3\frac{3}{4}$d. 3. The rate of interest is 7 dollars 30 cents per annum on 100 dollars. 4. The exact interest in English money is 2s. $10\frac{3}{4}$d. ·1240475.

VI.

1. See Art. 6. 2. The discount is £4 15s. $2\frac{6}{7}$d., and the interest of this discount for one year at 5 per cent. will be found to be 4s. $9\frac{1}{7}$d., which is the difference between £5, the interest, and £4 15s. $2\frac{6}{7}$d., the discount of £100 for one year. 3. The present worth is £222 4s. $5\frac{1}{3}$d., which, being deducted from £250, leaves the discount £27 15s. $6\frac{2}{3}$d. And the interest of £222 4s. $5\frac{1}{3}$d. at 5 per cent. at $2\frac{1}{2}$ years will be found equal to the discount of £250. 4. £673 4s. $11\frac{1}{4}$d. 5. £3 2s. 6d. per cent. 7. £733 19s. 3d. 8. £100 amounts to £$122\frac{1}{2}$ in five years, at $4\frac{1}{2}$ per cent. per annum. The sum required is £816 6s. $7\frac{18}{49}$d. 9. £1232 10s. $8\frac{3}{4}$d. 10. The interest due at the end of each half-year was £62 10s. The question requires the present worth of one-third of this sum, 3 months, one-third 2 months, and one-third 1 month, before the respective sums become due. 11. 625. 12. 18s. $6\frac{1}{4}$d. 13. £9 3s. 4d. 14. See question 2 above, and Art. 6.

VII.

1. £389 11s. $4\frac{1}{2}$d., amount when interest is payable yearly. £390 9s. $11\frac{1}{2}$d., amount when interest is payable half-yearly. 2. The rate £12 10s. $=\frac{1}{8}$ of £100. The amount is £1660 15s. $0\frac{3}{4}$d. exactly. 3. £129 6s. 4. The amount of £100 for 3 years at 5 per cent. compound interest is £115·7625. If £60 be considered the amount of which x pounds is the present worth, the sum will be found by a proportion to be £51 16s. $7\frac{1}{4}$d.

5. If the third year's interest be found, the interest for 195 days of the third year can be found by a proportion, and when this interest is added to the second year's amount, the sum will be the amount at the end of 2 years 195 days, from which, if the principal be subtracted, the remainder is the compound interest for that time. This is only an approximation, not the true result. See Art. 5, p. 6, note.

6. Here the interest of £410 is required for 10 periods, at the rate of $1\frac{1}{8}$ per cent. per annum for each period.

7. The difference between the simple and compound interest of £100 for 3 years at 4 per cent. is £1 19s. 8d. And the difference between the simple and compound interest of the sum required is 19s. By a proportion the sum is found to be £47 17s. $11\frac{3}{4}$d.

8. As the sum is not stated, let £100 be taken, and the amount of £100 in 2 years at compound interest is £108 16s. The requirement is to find at what rate simple interest £100 will amount to £108 16s. in 2 years.

9. Find the simple and compound interest of the given sum for 4 years, and reduce the ratio of the interests to its simplest form.

10. Find the successive yearly amounts.

11. £2027 1s. $6\frac{1}{4}$d.

12. The amount of £100 for 5 years at 5 per cent. is £123·812955; by a proportion may be found the present worth of £100 5 years hence.

13. See the last question. 14. To the first year's amount add £200, to make the principal for the second year, and so on in succession for the given number of years.

15. The legacy accumulates for 6 years at compound interest.

16. See question 14.

17. As the interest was not paid regularly, interest was also due on the interest unpaid for the whole of the period.

19. £500.

VIII.

1. The meaning is that £100 stock in the 3 per cent. consols is worth £88$\frac{3}{8}$ sterling.

£10,000 stock = £9000 sterling; let £100 stock = x sterling, thus $\frac{x}{9000}=\frac{100}{10000}$ and $x=$ £90 sterling, the price of £100 stock.

2. Here £100 stock = £95$\frac{3}{8}$ sterling; let £250 stock = x sterling, then $\frac{x}{95\frac{3}{8}}=\frac{250}{100}=\frac{5}{2}$ and $x=\frac{5}{2}\times 95\frac{5}{8}=$ £238 8s. 9d. sterling. Conversely, £100 stock = 95$\frac{3}{8}$ sterling; let x stock = £238 8s. 9d. sterling, and by a proportion the stock may be found.

Both in the purchase and sale of stock, the purchaser and seller always pay the commission $\frac{1}{8}$ per cent. to the broker. In the purchase of stock, the $\frac{1}{8}$ is an increase to the price of £100 stock; in the sale, it is a deduction from the sum received for £100 stock.

3. Here £96 sterling give £3 interest; let x sterling give £4 interest, then $\frac{x}{96}=\frac{4}{3}$, and $x=\frac{4\times 96}{3}=$ £125 6s. 8d.

4. If £100 sterling give x interest when invested in the 3 per cents at 96; then $\frac{x}{3}=\frac{100}{96}$ and $x=\frac{100\times 3}{96}=$ £3 2s. 6d. interest.

If £100 sterling give y interest when invested in the 3$\frac{1}{2}$ per cents at 103, then $\frac{x}{3\frac{1}{2}}=\frac{100}{103}$, and $x=\frac{100\times 2}{103\times 7}=$ £3 7s. 11$\frac{1}{2}$d. interest. The 3$\frac{1}{2}$ per cents give the greater interest.

5. He gains £150 by interest of his stock, and £150 by the transfer, so that he gains £300 on £4,600 sterling, which gives 6$\frac{12}{23}$ per cent.

6. The question is answered by finding which of the two stocks gives the greater interest for any fixed sum, as £100.

7. £971$\frac{47}{353}$ stock. 8. Consols give £3 2s. 6d. and insurance shares £3 6s. 1$\frac{1}{4}$d. per cent. interest.

9. Find the interest in each case, and reduce their ratio to the lowest terms.

10. £3920$\frac{7}{15}$ stock. 11. £60. 12. £89$\frac{63}{80}$. 13. £90.

14. £76$\frac{1}{2}$. 15. £31 17s. 6d. 16. By his investment he loses £49 5s. 9d. The portion of the half-yearly dividend is the interest of the stock from May 22 to July 6, at 3 per cent. per annum. 17. £400 stock. 18. £8 14s. 6d.

IX.

1. 106$\frac{19}{20}$ per cent. of railway stock. 2. The new stock £718 6s. 8$\frac{400}{709}$d., and the difference of the yearly interest of the old and new stock is 11s. nearly. 3. Income from stock in the 3 per cents is £381 8s. 6$\frac{6}{7}$d.: from stock in the 4 per cents, £466 13s. 4d. The difference of these, less 10d. in the pound for income tax, is £82 13s. 8$\frac{9}{14}$d. 4. £800 stock. 5. The 3$\frac{1}{2}$ per cents is the better investment by 5s. 0$\frac{1}{2}$d. interest yearly. 6. At 84$\frac{22}{55}$ per cent. 7. £1200 stock in the 3 per cent. consols is equivalent to £450 bank stock at 229$\frac{1}{3}$ per cent. 8. His income would be diminished by the transfer. 9. Railway stock gives £3 5s. yearly dividend on share of £100; 3 per cent. consols give £3 2s. 6$\frac{3}{4}$d. on £100 sterling. 10. Annual income increased by £17 2s. 4$\frac{1}{2}$d.

X.

1. The present worth of £400 due 2 years hence = £363$\frac{7}{11}$.
,, ,, £2100 ,, 8 ,, = 1500
Whole present worth of £2500 = £1863$\frac{7}{11}$.

The question is, In what time will £$1863\frac{7}{11}$ amount to £2500 at 5 per cent. simple interest?

If x be the number of years

$$\text{Then } £2500-1863\tfrac{7}{11}=£636\tfrac{4}{11} \text{ interest of } £1863\tfrac{7}{11} \text{ for } x \text{ years.}$$
$$\text{and } £96{\cdot}16\tfrac{2}{11} \quad ,, \quad 1 \text{ year.}$$
$$\text{Hence } \frac{x}{1}=\frac{636\frac{4}{11}}{93{\cdot}18\frac{2}{11}}=\frac{7000}{1025},$$
$$\text{or } x=6\tfrac{34}{41} \text{ years.}$$

The other three questions are solved in the same manner.

XI.

1. Here the whole stock is £20,000, which gives £5000. Let £8000 stock gain x, then $\frac{x}{5000}=\frac{5000}{20000}=\frac{2}{5}$ and $x=2000$, the first share, also £5000 − £2000 = £3000, the second share.

2. Here the amount of the 3 claims is £890, which pays £500. Let £$154\frac{1}{2}$ pay x, then $\frac{x}{500}=\frac{154\frac{1}{2}}{890}$, and $x=£86\frac{71}{89}$, payment for first claim. Similarly for the other claims.

3. The three shares are £$45276\frac{76}{199}$, £$54271\frac{71}{199}$, and £$80452\frac{52}{199}$.

4. Let A, B, C, D, denote the claims of A, B, C, D, respectively.

$$\text{Then } \frac{A}{B}=\frac{2}{3},\ \frac{B}{C}=\frac{4}{5},\ \frac{C}{D}=\frac{6}{7};\ \text{and let 1 denote } D\text{'s claim,}$$

$$\text{then } C\text{'s}=\frac{6}{7} \text{ of } D\text{'s}=\frac{6}{7},\ B\text{'s}=\frac{4}{5} \text{ of } C\text{'s}=\frac{4}{5} \text{ of } \frac{6}{7}=\frac{24}{35}, \text{ and } A\text{'s}=\frac{2}{3} \text{ of } B\text{'s}=\frac{2}{3} \text{ of } \frac{24}{35}=\frac{16}{35}.$$

$$\text{Sum of the claims}=1+\frac{6}{7}+\frac{24}{35}+\frac{16}{35}=3.$$

Let x_1 denote the share of assets due to D,

$$\text{then } \frac{x_1}{21000}=\frac{1}{3} \text{ and } x_1=£7000=D\text{'s share of assets.}$$
$$\frac{x_2}{21000}=\frac{1}{3}\times\frac{6}{7}=\frac{2}{7},\ x_2=£6000=C\text{'s share.}$$
$$\frac{x_3}{21000}=\frac{1}{3}\times\frac{24}{35}=\frac{8}{35},\ x_3=£4800=B\text{'s share.}$$
$$\frac{x_4}{21000}=\frac{1}{3}\times\frac{16}{35}=\frac{16}{105},\ x_4=£3200=A\text{'s share.}$$

5. In this question, the time forms an element of the effective cause. A's $=500\times1\frac{1}{2}=750$, B's $=1000\times1=1000$. The whole effective cause, 1750, produces £250.

Let 700 produce effect x.

$$\frac{x}{250}=\frac{750}{1750}=\frac{3}{7} \text{ and } x=£107\tfrac{1}{7}=A\text{'s share.}$$
$$\text{And } 250-107\tfrac{1}{7}=£142\tfrac{6}{7}=B\text{'s share.}$$

6. Their respective gains were £$246\frac{34}{71}$, £$507\frac{3}{71}$, and £$246\frac{34}{71}$.

7. The shares of A, B, C are £156 5s., £231 5s., and £310 respectively.

8. A's $400\times3=1200$ | B's $500\times5=2500$

$$\begin{array}{rr} 400\times3= & 1200 \\ 750\times9= & 6750 \\ \hline & 7950 \end{array} \qquad \begin{array}{rr} 500\times5= & 2500 \\ 450\times7= & 3150 \\ \hline & 5650 \end{array}$$

Here the sum of the effective causes is 13600, which produces £1020.

Let 7950 produce x,

$$\text{Then } \frac{x}{1020}=\frac{7950}{13600},\ x=£596\tfrac{17}{68}=A\text{'s share, and } £423\tfrac{51}{68}=B\text{'s share.}$$

9. *A* pays £20 and *B* £40. 10. *A* should receive £1014$\frac{6}{71}$, *B* £2464$\frac{54}{71}$, and *C* £1921$\frac{3}{71}$. 11. £$\frac{119}{883}$, £7$\frac{489}{883}$, £6$\frac{275}{883}$.

12. *Effective Causes.*

$$\begin{array}{lll}
A\text{'s } 500\times2=1000. & B\text{'s } 500\times3=1500. & C\text{'s } 500\times4=2000 \\
\underline{\ 50\ } & \underline{\ 75\ } & \underline{\ 100\ } \\
550\times2=1100 & 575\times3=1725 & 600\times4=2400 \\
\underline{\ 50\ } & \underline{\ 75\ } & \underline{\ 100\ } \\
600\times2=1200 & 650\times3=1950 & 700\times4=\underline{2800} \\
\underline{\ 50\ } & \underline{\ 75\ } & \qquad\qquad\ \ \underline{7200} \\
650\times2=1300 & 725\times3=\underline{2175} & \\
\underline{\ 50\ } & \qquad\qquad\ \ \underline{7350} & \\
700\times2=1400 & & \\
\underline{\ 50\ } & & \\
750\times2=\underline{1500} & & \\
\qquad\qquad\ \ \underline{7500} & &
\end{array}$$

The sum of the effective causes is 22050, which produces £661 10s.
The partners' shares are £225, £175 10s., and £226.

13. They receive £1500, £2000, and £2500 respectively. 14. *A* has to receive £524$\frac{13}{88}$, and *B* £720$\frac{75}{88}$.

XII.

1. £1 sterling = 25½ francs,
117 francs = 55 florins, then 1 franc $=\frac{55}{117}$ florin.
11 florins = 13 marks, then 1 florin $=\frac{13}{11}$ mark.
Hence £1 sterling = 25½ francs.
$=\frac{51}{2}\times\frac{55}{117}$ florins.
$=\frac{51}{2}\times\frac{55}{117}\times\frac{13}{11}$ marks.

2. Here 1½lb. cloves = 10lb. sugar; then 1lb. cloves $=\frac{20}{3}$lb. sugar.
17lb. sugar = 3lb. tea, and 1lb. sugar $=\frac{3}{17}$lb. tea.
Hence 1lb. cloves $=\frac{20}{3}\times\frac{3}{17}$lb. tea, and 100lb. cloves $=100\times\frac{20}{17}$lb. tea.

3. 112lb. pepper $=112\times\frac{13}{6}\times\frac{9}{38}\times\frac{63}{11}$lb. sugar at 5d. per pound. 4. £180. 5. 18 oxen.

XIII.

1. 10$\frac{1}{5}$s. a gallon. 2. 18s. 8d. 3. 8$\frac{7}{30}$d. per pound. 4. 2s. per pound. 5. 9$\frac{27}{37}$ per cent.

XIV.

1. 3$\frac{1}{8}$ per cent. 2. 20$\frac{5}{6}$ per cent. 3. 31¼ per cent. 4. £1388 17s. 9⅓d. 5. £1658 5s. 6. £150, £300, and £450 respectively. 7. £1296 17s. 6d.

XV.

1. When £100 is prime cost, gain is £25, and selling price is £125; let x be selling price when prime cost is 9d. The selling price is 10¼d. per pound. 2. At 8s. 7½d. to gain 15 per cent.: at 6s. 4½d. to lose 15 per cent. 3. 116⅔ per cent. per annum. 4. The remaining 30 yards must be sold for £17. 5. 3cwt. 2qrs. 25lb. 6oz. 13$\frac{47}{171}$drs. 6. One-eighth is equivalent to 12½ per cent. If selling at 5s. 4d. per pound gives a gain of 12½ per cent., what gain per cent. does the sale at 6s. per pound? 7. 9⅜ per cent. loss. 8. 18¾ per cent. gain. 9. The prime cost of £182 sales, with loss of 9 per cent., is £200. And 3 times this sum is £600, which at 7 per cent. gain gives £42, and if the loss on the first sale is to be made good, the goods bought for £400 must be sold for £478. 10. Loss of 8$\frac{4}{7}$ per cent. 11. £86$\frac{6}{11}$. 12. Here a capital of £142 10s. gains £21 3s. 11½d.; the requirement is, What does £100 gain? 13. £46$\frac{49}{86}$.

XVI.

1. He adds £35 to every £100 of the purchase, which makes £135 the selling price of £100 : and 10 per cent. for prompt payment on £135 is £13$\frac{1}{2}$, which deducted from £135 leaves £111$\frac{1}{2}$, leaving a gain of £11$\frac{1}{2}$ on £100 capital.

2. The present worth of £210 is £200$\frac{200}{209}$; his gain is the difference of 200$\frac{200}{209}$ and the prime cost, £170, which is £30$\frac{200}{209}$.

3. Prime cost £100, with gain £15, gives £115 selling price in 4 months, and in 12 months a sale of £345 is realised with £45 profit. Prime cost £100 with gain £20 gives selling price £120 in 6 months, and in 12 months a sale of £240 is realised with a gain of £40.

4. The present value of the bill is £600, which if put out to compound interest at 4$\frac{1}{2}$ per cent. per annum, payable quarterly, would amount to £627 9s. 2$\frac{1}{4}$d. at the end of the year. 5. Making the deduction by true discount, the cash price should be 11$\frac{10}{21}$s.

6. The manufacturer's cost of an article is £100.

Manufacturer adds 10 per cent. and sells to wholesale dealer for £110.

Wholesale dealer adds 15 per cent. to £110, and sells to retail dealer for £126$\frac{1}{2}$.

Retail dealer adds 30 per cent. to £126$\frac{1}{2}$, and sells to customer for £164$\frac{9}{20}$.

Thus what was made for £100 is sold at last for a profit of £64 9s., or £54 9s. to the two middle men.

XVII.

1. The 3 horses must be sold for £119 2s. 7d. to gain 10 per cent. on the prime cost of them.

2. Here £100 paper money = £80 gold, and hence £6 10s. gold is equivalent to £8 2s. 6d. paper money.

The purchase was made for	£6	10	0	paper money.
Deducting 5 per cent.	0	6	6	
Leaves sum to be paid in paper	6	3	6	
But value in gold offered was	8	2	6	in paper money.
Change in paper money received	£1	19	0	

3. Before the strike :	52 weeks' wages at 15s.	= 780s. 0d.
	2$\frac{1}{2}$ per cent. on yearly wages	19s. 6d.
	52 weeks' expense of living	= 760s. 6d.
After the strike :	52 weeks' wages at 17s.	= 884s. 0d.
	Expenses raised in ratio of 6 to 7	= 887s. 3d.
	Instead of saving, he is in debt	= 3s. 3d.

In other words, he was a poorer man by £1 2s. 9d. a year, after the strike than he was before it.

4. £540$\frac{20}{37}$ of 3 per cent. stock. 5. At 12s. 3$\frac{2}{11}$d. per gallon. 6. The gain is 25 per cent. ; the loss is 20 per cent. The difference arises from the fact that the gain and loss are calculated from different amounts of capital, the former from £88, the latter from £110.

7. £43$\frac{387}{287}$ gained. 8. Suppose the value to be £100, deducting banker's discount at 10 per cent. leaves £90 to be paid 6 months hence. If the £90 be paid 3 months before due, true discount ought to be deducted; if 3 months after, 3 months' interest ought to be added to the sum. 9. At 6$\frac{102}{175}$d. per oz. Avoirdupois.

XVIII.

1. The relative values of gold and silver are as 1869 to 124 ; or gold is in value a little more than 15 times the same weight of silver.

2. £114$\frac{1}{16}$.

3. The weight of the wedge of gold is found by dividing its value by the value of one ounce.

4. See question 7 above under XVII.

5. The relative values are as 725 to 572.

6. The proportion is 27 to 1.

XIX.

1. A quarter of corn in France costs 38$\frac{47}{80}$ shillings.

2. One cubic foot of water weighs $6\times20\times14$ pounds Avoirdupois, and $\frac{1000\times1000}{1131}$ kilogrammes.

3. £256 7s. 10$\frac{2}{7}$d. 4. 21$\frac{1}{2}$ feet. 5. £731 1s. 4$\frac{1}{4}$d. 6. £50 1s. 8$\frac{3}{4}$d.

XX.

1. 70 days. 2. 27$\frac{35269}{53406}$ weeks. 3. 45 men to be added. 4. 156$\frac{16}{35}$ days. 5. £31$\frac{1}{9}$. 6. £125. 7. £50 10s. 8. 108 days. 9. 98 labourers.

XXI.

1. 24 days. 2. 11$\frac{3}{7}$ hours. 3. 19·72 English miles. 4. $\frac{1339}{52800}$ of one inch, which is greater than $\frac{1}{39}$, but less than $\frac{1}{40}$. 5. 46$\frac{23}{31}$ feet. Euclid VI. 4.

6. The equatorial diameter is equal to $\frac{299.33\times7899.114}{283.33}$ miles.

7. The ratio of one metre to one foot is 3·28035.

8. If lunarian eyes have the same powers of vision as sublunarian eyes, the apparent disks will be proportional to the squares of the diameters of the earth and moon, and one will appear 16 times as large as the other.

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THE PROPERTIES AND CONSTRUCTION OF LOGARITHMS.

Art. 1. Def. 1. In the equation $u = a^x$, where a is a constant number greater than unity, and u any natural number, the index x is defined to be the logarithm of the number u to the base a.

The notation assumed to denote "the logarithm of the number u to the base a" is $\log_a u$, so that $x = \log_a u$, and the equation $u = a^x$ may be written $u = a^{\log_a u}$.

Def. 2. The base of any system of logarithms is any fixed number which, being raised to the powers denoted by the logarithms, produces the successive natural numbers.

Def. 3. A system of logarithms is a series of the successive values of x derived from the equation $u = a^x$, when the natural numbers 0, 1, 2, 3, 4, &c., are successively substituted for u, the same base a being preserved.[1]

---

[1] Logarithms may be defined to be, as in fact they are, a series of numbers in Arithmetical progression which increase by a common difference, corresponding to another series in Geometrical progression which increase by a common multiplier. For example, let 10 be made the base,

$$\begin{array}{llllllll} \text{If } 0, & 1, & 2, & 3, & 4, & 5, \;.\;. & n & \text{be a series in A. P.} \\ \text{and } 10^0, & 10^1, & 10^2, & 10^3, & 10^4, & 10^5, \ldots & 10^n & \multirow{2}{*}{$\left.\begin{array}{l}\\ \\ \end{array}\right\}$ be a corresponding series in G. P.} \\ \text{or } \quad 1, & 10, & 100, & 1000, & 10000, & 100000 \ldots & 10^n & \end{array}$$

Then the numbers 0, 1, 2, 3, 4, &c., are the logarithms of the series of numbers 1, 10, 100, 1000, 10000 &c., respectively, to the base 10.

Hence it is obvious that a negative number cannot be assumed as the base of any system of logarithms; for the odd powers of a negative number are negative, and the even powers are positive, and consequently they are not subject to the law of continuity in producing in order all the natural numbers.

This definition of a system of logarithms suggests a method of finding the logarithms of all the intermediate numbers; for the Arithmetic mean between any two consecutive terms of the Arithmetic series will be the logarithm of the Geometric mean of the two corresponding terms of the Geometric series:

Thus, the A. mean between 0 and 1 is ·5,
and the G. mean between 1 and 10 is 3·1622777;
Hence ·5 is the logarithm of 3·1622777.
Again the A. mean between 1 and 2 is 1·5,
and the G. mean between 10 and 100 is 31·6227766;
Hence 1·5 is the logarithm of 31·6227766;
and so on for successive mean proportionals.

Next, the A. means can be found between every two consecutive terms of the A. series 0, ·5, 1, 1·5, &c., and the G. means between every two corresponding terms of the G. series 1, 3·1622777, 10, 31·6227766, 100, &c.; and so continuing the same process, may be found the logarithms of all numbers, but at very great expense of time and labour.

A system of logarithms may be calculated to any base except unity, and hence there may be an indefinite number of systems of logarithms, according to the different assumptions made for the bases. There are, however, only two systems of logarithms used by mathematicians, one for shortening numerical calculations and the other in analytical reasonings.

The following consequences may be shown to arise from the equation $u = a^{\log_a u}$:—

The logarithm of 1 is 0, or $\log_a 1 = 0$; and the logarithm of the base is 1, or $\log_a a = 1$.

If $\log_a u$ be positive, and assume successively and continuously all possible values from 0 to $+\infty$, it is obvious that u will receive all values from 1 to ∞.

If $\log_a u$ be negative, and assume all possible values from 0 to $-\infty$, u will receive all values from 1 to 0.

Hence, as $\log_a u$ changes continuously from $+\infty$ to $-\infty$, u changes continuously from $+\infty$ to 0, and consequently produces all the positive natural numbers.

If the base a be 10 and remain constant, and u be made to assume successively 1, 2, 3, 4, &c., the corresponding values of x in the equation $u = 10^x$, when computed and registered will form a table of that system of logarithms whose base is 10.

2. Prop. *To find the logarithm of the product of two numbers.*

Here $u_1 = a^{\log_a u_1}$, and $u_2 = a^{\log_a u_2}$, by def.

$$\therefore u_1 \,.\, u_2 = a^{\log_a u_1} \,.\, a^{\log_a u_2} = a^{\log_a u_1 + \log_a u_2}$$

And $\log_a \{u_1 \,.\, u_2\} = \log_a u_1 + \log_a u_2$ by def.

Or, the logarithm of the product of two numbers, is equal to the sum of the logarithms of the numbers themselves.

Cor. In a similar way it may be shewn that the

$$\log_a \{u_1 \,.\, u_2 \,.\, u_3 \,.\, \ldots\} = \log_a u_1 + \log_a u_2 + \log_a u_3 + \ldots.$$

Or that the logarithm of the product of any number of factors, is equal to the sum of the logarithms of the several factors.

3. Prop. *To find the logarithm of the quotient of two numbers.*

Here $u_1 = a^{\log_a u_1}$, and $u_2 = a^{\log_a u_2}$ by def.

$$\text{Then } \frac{u_1}{u_2} = \frac{a^{\log_a u_1}}{a^{\log_a u_2}} = a^{\log_a u_1 - \log_a u_2}.$$

$$\therefore \log_a \left\{ \frac{u_1}{u_2} \right\} = \log_a u_1 - \log_a u_2.$$

Or, the logarithm of a quotient, is equal to the difference arising from subtracting the logarithm of the divisor from the logarithm of the dividend.

Cor. $\log_a \left\{ \frac{1}{u_2} \right\} = \log_a u_1 - \log_a u_2 = -\{\log_a u_2 - \log_a u_1\} = -\log_a \left\{ \frac{u_2}{u_1} \right\}.$

Or, the logarithm of any fraction is equal to the logarithm of its reciprocal taken negatively.[1]

4. PROP. *To find the logarithm of any power of a number.*

Here $u = a^{\log_a u}$ by def.

Then raising each to the n^{th} power.

$$a^n = a^{n \log_a u}.$$

$$\therefore\ \log_a\{u^n\} = n \log_a u.$$

Or, the logarithm of any power of a number, is equal to the product of the logarithm of the number and the index of the power.

5. PROP. *To find the logarithm of any root of a number.*

Here $u = a^{\log_a u}$ by def.

Then $u^{\frac{1}{n}} = \{a^{\log_a u}\}^{\frac{1}{n}} = a^{\frac{1}{n}\log_a u}.$

And $\log_a\{u^{\frac{1}{n}}\} = \frac{1}{n}\log_a u.$

Or, the logarithm of any root of a number, is equal to the quotient arising from dividing the logarithm of the number by the index of the root.

Hence it appears that if a table of the logarithms of the natural numbers be arranged in order; by means of them can be performed the operations of multiplication, division, involution and evolution of all numbers within the limits of the table. Thus, if one number is to be multiplied or divided by another, by taking their logarithms from the table, and adding or subtracting them, and then by finding in the table the number whose logarithm is equal to the sum or difference, the product or quotient of the two numbers is found. And the power or root of a number is found by taking the logarithm of the number from the table, and multiplying or dividing it by the index of the power or root, and then by finding in the table the number whose logarithm is equal to this product or quotient, the power or root of the proposed number is determined. Thus, by the aid of a table of logarithms, the arithmetical operations of multiplication and division may be effected by addition and subtraction: and those of involution or evolution by multiplying or dividing the logarithm by the index of the power or root. These are the advantages of logarithms in effecting numerical computations.

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[1] Care must be taken not to confound the expressions $\log_a \left\{\frac{u_1}{u_2}\right\}$ and $\frac{\log_a u_1}{\log_a u_2}$: the former being the logarithm of the quotient of two numbers, which has been shewn equal to the difference arising from subtracting the logarithm of the divisor u_2 from the logarithm of the dividend u_1; while the latter is the quotient arising from dividing the logarithm of u_1 by the logarithm of u_2.

DEF. The integral part of a logarithm is named its characteristic, the decimal part its mantissa.[1]

In all arithmetical computations by logarithms, the mantissa is always *positive,* but the characteristic may be *positive* or *negative.*

6. PROP. *To explain the advantages of that system of logarithms whose base is* 10, the same as the radix of the scale of notation.

By considering $\log_{10}(10^n.u)$ and $\log_{10}\left\{\frac{u}{10^n}\right\}$;

$$\log_{10}\{10^n.u\} = \log_{10} 10^n + \log_{10}u = n \log_{10}10 + \log_{10}u = n + \log_{10}u,$$

$$\text{and } \log_{10}\left\{\frac{u}{10^n}\right\} = \log_{10}u - \log_{10}10^n = -n \log_{10}10 + \log_{10}u = -n + \log_{10}u;$$

That is, the $\log_{10}\{10^n.u\}$ and $\log_{10}\left\{\frac{u}{10^n}\right\}$ are found from $\log_{10}u$, by simply *increasing* or *diminishing* the characteristic of $\log_{10}u$ by n.

Hence, the logarithms of all numbers consisting of the same significant figures, whether integral, decimal, or partly integral and partly decimal, have the same mantissa; the only difference being in the value of the characteristic.

7. PROP. *To find the law of the characteristics of that system of logarithms whose base is* 10.[2]

Let any integral number u consist of n digits.

It lies between 10^{n-1} and 10^n;

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[1] The word *mantissa* appears to be a Tuscan word, formerly employed in commerce, and meaning over-measure or over-weight, "additamentum quod ponderi adjicitur."

The following logarithms of the prime numbers less than 100 are here given to enable the student to obtain numerical results in the exercises. In the printed tables of logarithms, the characteristics are omitted, and only the decimal parts are given without the decimal point. Of the Mathematical tables published by Dr. Hutton, one table calculated to seven places of decimals contains the logarithms of the natural numbers from 1 to 100,000. In the table published by Mr. Babbage, the logarithms of the numbers are extended from 1 to 108,000, and very great care was taken by Mr. Babbage to secure the accuracy of them.

Nos.	Logarithms.	Nos.	Logarithms.	Nos.	Logarithms.
2	3010300	29	4623980	61	7853298
3	4771213	31	4913617	67	8260748
7	8450980	37	5682017	71	8512583
11	0413927	41	6127839	73	8633229
13	1139434	43	6334685	79	8976271
17	2304489	47	6720979	83	9190781
19	2787536	53	7242759	89	9493900
23	3617278	59	7708520	97	9867717

[2] If the number assumed for the base of a system of logarithms be the same as the radix of the system of notation employed, a great advantage arises; as in the system of notation whose radix is 10, the mantissa of any number composed of the same digits will have the same mantissa, whether the number be integral, or decimal, or

and therefore the logarithm of u lies between $n-1$ and n, and consequently consists of $n-1$ units increased by some decimal: that is, the characteristic of $\log_{10} u$ is $n-1$.

Next, let u' be a decimal having $n-1$ ciphers between the decimal point and the first significant figure.

This decimal u' lies between $\frac{1}{10^{n-1}}$ and $\frac{1}{10^{n}}$, or $10^{-(n-1)}$ and 10^{-n},

$\therefore$ the logarithm of u' lies between $-(n-1)$ and $-n$, and consequently consists of $-n$, increased by some positive decimal part: that is, the characteristic of $\log_{10} u'$ is $-n$.

Hence the general rule. For numbers wholly or partly integral, the characteristic is always less by unity than the number of integral places of which the number consists: and for decimals, the characteristic is the number (taken negative) which expresses the distance of the first significant figure of the decimal from the place of units.

8. PROP. *The logarithm of a number less than 1, being negative, can always be expressed so that its mantissa shall be positive, and only its characteristic negative.*[1]

Let u be a number less than 1,
n the characteristic, m the mantissa of its logarithm.

Then $\log_{10} u = -(n+m) = -n - m + 1 - 1$
$= -(n+1) + (1-m)$ of which $1-m$ is positive.

Hence a logarithm wholly negative may be transformed into one whose characteristic only is negative, by increasing the negative characteristic by 1 and replacing the mantissa by its arithmetic complement, or its defect from 1.

And conversely. A logarithm whose characteristic only is negative, may be transformed into a logarithm wholly negative by diminishing the characteristic by 1, and replacing the mantissa or decimal part by its arithmetical complement.

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partly integral and partly decimal, as will be seen in the logarithms of the numbers composed of the significant digits 6375.

Numbers.	Logarithms.	Numbers.	Logarithms.
6375	3·8044802	637·5	2·8044802
63750	4·8044802	63·75	1·8044802
637500	5·8044802	6·375	0·8044802
6375000	6·8044802	·6375	$\bar{1}$·8044802
63750000	7·8044802	·06375	$\bar{2}$·8044802
637500000	8·8044802	·006375	$\bar{3}$·8044802

[1] Ex. 1. $\mathrm{Log}_{1}\left\{\frac{1}{2}\right\} = \log_{10} 1 - \log_{10} 2 = -\log_{10} 2 = -\cdot 3010300$
$= -1 + (1 - \cdot 3010300)$
$= \bar{1}\cdot 6989700$, a logarithm with its mantissa positive and its characteristic negative.

Ex. 2. $\mathrm{Log}_{10}\left\{\frac{5}{9}\right\} = \log_{10} 5 - \log_{10} 9$, which is wholly negative.
$= (1 + \log_{10} 5) - 1 - \log_{10} 9 = (\log_{10} 10 + \log_{10} 5) - 1 - \log_{10} 9$
$= \bar{1} + (\log_{10} 50 - \log_{10} 9)$.

In practice, a negative logarithm is always expressed so that its *characteristic only is negative*, and the negative sign is placed over the characteristic, which is separated by a point from the decimal part, which is positive.

When negative logarithms are expressed in this manner, a proper distinction must be made between the contrary signs of the characteristic and the mantissa, in the operations of their addition, subtraction, and multiplication by any number. Also, in dividing a logarithm whose characteristic is negative by any number, the negative characteristic must be made exactly divisible, by adding to it the least negative number which makes it so divisible, and this process must be corrected by the addition of an equal positive integral number to the mantissa.

In the system of logarithms whose base is 10, since the logarithms of all numbers not exact powers of 10, are incommensurable, their values can be obtained only approximately by decimals.

Hence the logarithms of all numbers greater than 1, not exact powers of 10, will consist of positive numbers partly integral and partly decimal, except the logarithms of numbers less than 10: and the logarithms of all numbers less than 1 will consist of negative numbers partly integral and partly decimal.

As the characteristics of the logarithms of this system can always be found by inspection, they are omitted, and the mantissæ only are registered in the tables.

9. Prop. *To find the relation between the logarithms of the same number, but of different bases.*

Let x, z denote the logarithms of the number u to the bases a, c, respectively.

Then $u = a^x$ and $x = \log_a u$; $u = c^z$, and $z = \log_c u$.

Whence $a^x = c^z$, and, taking the logarithms of these equals to base c.

Then $x \log_c a = z \log_c c = z$.

$$\text{but since } x = \log_a u \text{ and } z = \log_c u$$

$$\therefore \log_a u \,.\, \log_c a = \log_c u$$

$$\text{and } \log_a u = \frac{\log_c u}{\log_c a}$$

That is; the logarithm of any number u to base a, is equal to the logarithm of the same number u to base c, multiplied by the reciprocal of the logarithm of a to base c.

This multiplier is called the *modulus* of the system of logarithms whose base is a. For the logarithms of all numbers calculated to base c are converted into logarithms of the same number to base a by multiplying each logarithm by $\frac{1}{\log_c a}$.

Cor. If $u = c$, then $\log_a c = \frac{\log_c c}{\log_c a}$, and $\therefore \log_a c \,.\, \log_c a = \log_c c = 1$.

10. **Prop.** *To find the modulus for transforming Napier's logarithms to logarithms whose base is* 10.

Since it has been proved generally that $\log_a u = \dfrac{\log_c u}{\log_c a}$

Let $a = 10$ and $c = e$ the base of Napier's system,

$$\text{then } \log_{10} u = \frac{\log_e u}{\log_e 10} = \frac{1}{\log_e 10} \cdot \log_e u.$$

Hence the modulus required is $\dfrac{1}{\log_e 10}$; and logarithms of Napier's system can be transformed to logarithms whose base is 10 by multiplying each of Napier's logarithms by the modulus $\dfrac{1}{\log_e 10}$.

11. **Prop.** *To express the number u in a series, in terms of the base and of the logarithm of the number; or in the equation* $u = a^x$, *to expand* a^x *in a series of ascending powers of* x.[1]

Here $u = a^x = \{1 + (a - 1)\}^x$

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[1] The following method of the expansion of a^x by means of an indeterminate index, is taken from La Grange's Calcul de Fonctions.

Let t be any number supposed to vary its value.

Then $a^t = \{1 + (a-1)\}^t$

$$= 1 + t\,(a-1) + \frac{t\,(t-1)}{1.2} \cdot (a-1)^2 + \frac{t\,(t-1)\,(t-2)}{1.2.3} \cdot (a-1)^3 + \ldots.$$

$$= 1 + t\,(a-1) + \left\{\frac{t^2}{2} - \frac{t}{2}\right\} . (a-1)^2 + \left\{\frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{3}\right\} . (a-1)^3 + \ldots.$$

$$= 1 + t\left\{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots.\right\} + B\,t^2 + C\,t^3 + \ldots.$$

by collecting the co-efficients of t, and putting B, C, &c., which are functions of a, for the co-efficients of t^2, t^3, &c.,

And if $(a-1) - \dfrac{(a-1)^2}{2} + \dfrac{(a-1)^3}{3} - \ldots. = A$, and $P t^2 = t^2\,(B + C t + \ldots.)$

$$a^t = 1 + At + Pt^2 = 1 + (A + Pt)t$$

$$\therefore\ a^x = a^{\frac{tx}{t}} = \{a^t\}^{\frac{x}{t}} = \{1 + (A + Pt)t\}^{\frac{x}{t}}$$

$$= 1 + \frac{x}{t}(A + Pt)\,t + \frac{\frac{x}{t}\left(\frac{x}{t} - 1\right)}{1\cdot2} \cdot (A + Pt)^2\,t^2$$

$$+ \frac{\frac{x}{t}\left(\frac{x}{t} - 1\right)\left(\frac{x}{t} - 2\right)}{1.2.3} \cdot (A + Pt)^2\,t^3 + \ldots.$$

$$= 1 + x\,(A + Pt) + \frac{x\,.\,(x - t)}{1.2} \cdot (A + Pt)^2$$

$$+ \frac{x\,(x - t)\,(x - 2t)}{1.2.3}\,(A - Pt)^3 + \ldots.$$

Now this is true whatever may be the value of t, for a^x is obviously independent of t. Let $t = o$

$$\therefore\ a^x = 1 + Ax + \frac{A^2\,x^2}{1.2} + \frac{A^3\,x^3}{1.2.3} + \ldots.$$

$$\text{in which } A = (a - 1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \ldots.$$

$$=1+x\,(a-1)+\frac{x\,(x-1)}{1.2}.(a-1)^2+\frac{n\,(n-1)(n-2)}{1.2.3}.(a-1)^3+\&c.$$

$$=1+\left\{(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.\right\}x+Bx^2+Cx^3+\&c.,$$

by finding the terms of the coefficient of x, and putting for the coefficients of x^2, x^3, &c., B, C, D, which involve numerical quantities and powers of a only.

And by putting A for $(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.$

$$a^x=1+Ax+Bx^2+Cx^3+Dx^4+\&c.$$

Now this expression for a^x is true, whatever value may be assigned to x, let x become $x+z$.

Then $a^{x+z}=1+A\,(x+z)+B\,(x+z)^2+C\,(n+z)^3+\&c.$

but $a^{x+z}=a^x\,.\,a^z=a^z\{1+Ax+Bx^2+Cx^3+\&c.\}$

Equating the coefficients of x in these identical values of a^{x+z},

$$A+2\,Bz+3\,Cz^2+4\,Dz^3+\&c.=Aa^z=A\,\{1+Az+Bz^2+Cz^3+\&c.\}$$

And, lastly, equating the coefficients of the same powers of z in these identical series,

$$2B=A^2,\ 3C=AB,\ 4D=AC,\ \&c.$$

Whence $B=\frac{A^2}{1.2}$, $C=\frac{AB}{3}=\frac{A^3}{1.2.3}$, $D=\frac{AC}{4}=\frac{A^4}{1.2.3.4}$, &c.

$$\therefore\ u=a^x=1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\frac{A^4x^4}{1.2.3.4}+\&c.$$

$$\text{in which } A=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.$$

Cor. If e denote that value of a in the equation $u=a^x$ which makes the series for A equal to unity, then $u=e^x$ and $\log_e u=x$, and the series for u or e^x becomes

$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c.$, which is the number u expressed in terms of x the logarithm of e; and making $x=1$, $e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.=2.7182818\ldots$ which is the numerical value of the base e of Napier's system of logarithms.[1]

12. Prop. *Shew that the base of the Napierian system of logarithms is incommensurable.*

$$\text{Here } e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots.$$

$\therefore\ e-2=\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots.$ a series which is evidently less than the series $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots$

---

[1] It is seldom the value of the base of Napier's system of logarithms is required in any calculations beyond seven places of decimals. The following is the value of e calculated to twenty places of decimals: $e=2{\cdot}718281,828459,045235,36\ldots..$

which is a geometrical series whose sum is 1.

Hence e is greater than 2 but less than 3.

Again, the value of e cannot be expressed by any finite fraction $\frac{m}{n}$, for then we should have

$$\frac{m}{n} - 2 = \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \ldots + \frac{1}{1.2 \ldots n}$$
$$+ \frac{1}{1.2 \ldots n(n+1)} + \ldots .$$

Multiplying both sides by $1 . 2 . 3 \ldots n$.

$$\therefore m.1.2.3 \ldots (n-1) - 1.2^2.3 \ldots n = 3.4 \ldots n + 4.5 \ldots n + \ldots$$
$$\ldots + 1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots$$

Now the left side of this equation is an integer; the right side must also be an integer.

But the series $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots$ is less than the sum of the series $\frac{1}{n+1} + \frac{1}{(n+1)^2} +$ &c. . . . which is equal to $\frac{1}{n}$.

$\therefore$ The right side cannot be a whole number.

Hence there is no integer or finite fraction by which the exact value of e can be expressed.

13. Prop. *To find the logarithm of any number in a series in terms of the base and of the number itself; or to find x from the equation $u = a^x$, in terms of u and a.*[1]

---

[1] The following is from La Grange's Calcul de Fonctions :—

Here $u = a^x$ and $x = \log_a u$.

$$\text{Now } 1 + (u-1) = \{1 + (a-1)\}^x$$
$$\therefore \{1 + (u-1)\}^t = \{1 + (a-1)\}^{tx}$$

Expanding each side by the binomial theorem,

$$1 + t(u-1) + \frac{t(t-1)}{1.2}.(u-1)^2 + \frac{t(t-1)(t-2)}{1.2.3}.(u-1)^3 + \text{\&c.}$$
$$= 1 + tx(a-1) + \frac{tx(tx-1)}{1.2}.(a-1)^2 + \frac{tx(tx-1)(tx-2)}{1.2.3}.(a-1)^3 + \text{\&c.}$$

subtracting 1 from each side and dividing every term by t,

$$\therefore (u-1) + \frac{(t-1)}{1.2}.(u-1)^2 + \frac{(t-1)(t-2)}{1.2.3}.(u-1)^3 + \text{\&c.}$$
$$= x\left\{(a-1) + \frac{(tx-1)}{1.2}.(a-1)^2 + \frac{(tx-1)(tx-2)}{1.2.3}.(a-1)^3 + \text{\&c.}\right.$$

This equation being true whatever be the value of t. Let $t = o$

$$\therefore (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \text{\&c.} = x\left\{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}\right\}$$

$$\therefore x \text{ or } \log_a u = \frac{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \text{\&c.}}{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}}$$

which is the logarithm of the number u in terms of u and the base a.

Since $u = a^x$, in which $x = \log_a u$.

And $a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} +$ &c., is true whatever values may be assigned to a and x.

Let x, a, become respectively z, u, and A_1 the new value of A.

$$\text{Then } u^z = 1 + A_1 z + \frac{A_1^{\,2} z^2}{1.2} + \frac{A_1^{\,3} z^3}{1.2.3} + \frac{A_1^{\,4} z^4}{1.2.3.4} + \text{\&c.}$$

$$\text{in which } A_1 = (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \frac{(u-1)^4}{4} + \text{\&c.}$$

But since $u = a^x$,

$$\therefore\ u^z = (a^x)^z = a^{xz} = 1 + Axz + \frac{A^2x^2z^2}{1.2} + \frac{A^3x^3z^3}{1.2.3} + \text{\&c.}$$

But it has been shown above that

$$u^z = 1 + A_1 z + \frac{A_1^{\,2} z^2}{1.2} + \frac{A_1^{\,3} z^3}{1.2.3} + \text{\&c.}$$

Hence, equating the coefficients of z in these identical series,

$$\therefore\ Ax = A_1,$$

$$\text{and } x \text{ or } \log_a u = \frac{A_1}{A} = \frac{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \frac{(u-1)^4}{4} + \text{\&c.}}{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}}$$

which is the logarithm of the number u, to base a, expressed in terms of the number itself and the base.

It is obvious that this expression is not adapted for the calculation of the logarithms of natural numbers either to the base 10, or to the base of Napier's system, because the series do not converge for numbers greater than 2.

14. Prop. *To find a converging series for calculating Napier's logarithms.*

$$\text{It has been proved, that } \log_a u = \frac{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \text{\&c.}}{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \text{\&c.}}$$

whatever may be the assumed values of a and u.

Now in the Napierian system the base is denoted by e; writing e for a.

$$\therefore\ \log_e u = \frac{(u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \text{\&c.}}{(e-1) - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \text{\&c.}}$$

But in Napier's system e is of such a value that the series

$$(e-1) - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \text{\&c.} = 1$$

Hence the expression becomes

$\log_e u = (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} -$ &c., which is the Napierian logarithm of u expressed in terms of the number u only.

This series will be found *divergent* for all values of u greater than 2, and is, therefore, not adapted for the calculation of Napier's logarithms. It must therefore be transformed into a series which shall be *convergent* for all numbers.

To effect this object, for u write $1+m$ and $1-m$ respectively,

$$\text{Then } \log_e(1+m) = m - \frac{m^2}{2} + \frac{m^3}{3} - \frac{m^4}{4} + \text{\&c.}$$

$$\log_e(1-m) = -m - \frac{m^2}{2} - \frac{m^3}{3} - \frac{m^4}{4} - \text{\&c.}$$

$$\therefore \log_e(1+m) - \log_e(1-m) \text{ or } \log_e\left\{\frac{1+m}{1-m}\right\} = 2\{m + \frac{m^3}{3} + \frac{m^5}{5} + \text{\&c.}\}$$

$$\text{next let } \frac{1+m}{1-m} = \frac{u}{v}, \text{ then } m = \frac{u-v}{u+v},$$

$$\therefore \log_e\left\{\frac{u}{v}\right\} \text{ or } \log_e u - \log_e v = 2\left\{\frac{u-v}{u+v} + \frac{1}{3}\cdot\left(\frac{u-v}{u+v}\right)^3 + \frac{1}{5}\cdot\left(\frac{u-v}{u+v}\right)^5 + \text{\&c.}\right\}$$

$$\text{and } \log_e u = \log_e v + 2\left\{\frac{u-v}{u+v} + \frac{1}{3}\cdot\left(\frac{u-v}{u+v}\right)^3 + \frac{1}{5}\cdot\left(\frac{u-v}{u+v}\right)^5 + \text{\&c.}\right\}$$

which is a series converging for all numbers greater than unity, and is therefore adapted for calculating Napier's logarithms of the natural numbers.[1]

15. Prop. *To calculate to seven places of decimals the numerical value*

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[1] By means of this formula, the logarithms of the prime numbers may be calculated to the base e, and the logarithms of the composite numbers can be found from the logarithms of the prime numbers.

$$\text{For generally } \log_e u = \log_e v + 2\left\{\frac{u-v}{u+v} + \frac{1}{3}\left(\frac{u-v}{u+v}\right)^3 + \frac{1}{5}\left(\frac{u-v}{u+v}\right)^5 + \text{\&c.}\right\}$$

$$\text{Let } u=2,\ v-1 : \log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \text{\&c.}\right\}$$

$$u-3,\ v=2 : \log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} + \text{\&c.}\right\}$$

$$u=5,\ v=4 : \log_e 5 = \log_e 4 + 2\left\{\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5} + \text{\&c.}\right\}$$

$$u=7,\ v=6 : \log_e 7 = \log_e 6 + 2\left\{\frac{1}{13} + \frac{1}{3}\cdot\frac{1}{13^3} + \frac{1}{5}\cdot\frac{1}{13^5} + \text{\&c.}\right\}$$

and so on for all the succeeding prime numbers.

As composite numbers are composed of the products of two or more prime numbers, the logarithms of all composite numbers may be found from the logarithms of their prime factors by means of Articles 2, 4, on p. 2 and p. 3.

Thus, $\log_e 4 = \log_e 2^2 = 2\log_e 2$.
$\log_e 6 = \log_e(2.3) = \log_e 2 + \log_e 3$.
$\log_e 8 = \log_e 2^3 = 3\log_e 2$.
$\log_e 9 = \log_e 3^2 = 2\log_e 3$.
$\log_e 10 = \log_e(2.5) = \log_e 2 + \log_e 5$.
$\log_e 12 = \log_e(2^2.3) = 2\log_e 2 + \log_e 3$.
and so for succeeding composite numbers.

of the modulus for transforming Napier's logarithms to logarithms whose base is 10.[1]

The modulus $M = \frac{1}{\log_e 10}$,

$$\text{and } \log_e 10 = \log_e (2 \times 5) = \log_e 2 + \log_e 5.$$

$$\text{Now } \log_e u = \log_e v + 2\left\{\frac{u-v}{u+v} + \frac{1}{3}\left(\frac{u-v}{u+v}\right)^3 + \frac{1}{5}\left(\frac{u-v}{u+v}\right)^5 + \&\text{c.}\right\}$$

Let $u = 2$, $v = 1$,

---

[1] The calculation of $\log_e 2$ and $\log_e 5$ may be thus exhibited :—

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \&\text{c.}\right\}$$

$$\frac{1}{3} = \cdot 333333333$$

$$\frac{1}{3^3} = \frac{1}{9}\cdot\frac{1}{3} = \cdot 037037037 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{3}\cdot\frac{1}{3^3} = \cdot 012345678$$

$$\frac{1}{3^5} = \frac{1}{9}\cdot\frac{1}{3^3} = \cdot 004115226 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{5}\cdot\frac{1}{3^5} = \cdot 000823045$$

$$\frac{1}{3^7} = \frac{1}{9}\cdot\frac{1}{3^5} = \cdot 000457247 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{7}\cdot\frac{1}{3^7} = \cdot 000065321$$

$$\frac{1}{3^9} = \frac{1}{9}\cdot\frac{1}{3^7} = \cdot 000050805 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{9}\cdot\frac{1}{3^9} = \cdot 000005645$$

$$\frac{1}{3^{11}} = \frac{1}{9}\cdot\frac{1}{3^9} = \cdot 000005645 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{11}\cdot\frac{1}{3^{11}} = \cdot 000000513$$

$$\frac{1}{3^{13}} = \frac{1}{9}\cdot\frac{1}{3^{11}} = \cdot 000000627 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{13}\cdot\frac{1}{3^{13}} = \cdot 000000048$$

$$\frac{1}{3^{15}} = \frac{1}{9}\cdot\frac{1}{3^{13}} = \cdot 000000069 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{15}\cdot\frac{1}{3^{15}} = \cdot 000000004$$

$$\frac{1}{3^{17}} = \frac{1}{9}\cdot\frac{1}{3^{15}} = \cdot 000000007 \quad \ldots \quad \ldots \quad \ldots \quad \therefore \frac{1}{17}\cdot\frac{1}{3^{17}} = \cdot 000000000$$

$$\cdot 346573587$$

$$2$$

$$\log_e 2 = \cdot 693147174$$

$$\log_e 5 = \log_e 4 + 2\left\{\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5} + \&\text{c.}\right\}$$

$$\frac{1}{9} = \cdot 111111111$$

$$\frac{1}{9^3} = \frac{1}{9} \times \cdot 012345678 = \cdot 001371742 \quad \ldots \quad \ldots \quad \therefore \frac{1}{3}\cdot\frac{1}{9^3} = \cdot 000457247$$

$$\frac{1}{9^5} = \frac{1}{9} \times \cdot 000152415 = \cdot 000016935 \quad \ldots \quad \ldots \quad \therefore \frac{1}{5}\cdot\frac{1}{9^5} = \cdot 000003387$$

$$\frac{1}{9^7} = \frac{1}{9} \times \cdot 000001881 = \cdot 000000209 \quad \ldots \quad \ldots \quad \therefore \frac{1}{7}\cdot\frac{1}{9^7} = \cdot 000000029$$

$$\cdot 111571772$$

$$2$$

$$\cdot 223143544$$

$$\log_e 4 = 2 \log_e 2 = 1 \cdot 386294348$$

$$\therefore \log_e 5 = 1 \cdot 609437892$$

$\text{Log}_e 10 = 2\cdot 302585,092994,045684,017991,\ldots$ to twenty-four places of decimals.

$$\therefore\ \log_e 2 = \log_e 1 + 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \&c.\right\} = {\cdot}693147174\ldots$$

Let $u = 5$, $v = 4$,

$$\log_e 5 = \log_e 4 + 2\left\{\frac{1}{9} + \frac{1}{3}\cdot\frac{1}{9^3} + \frac{1}{5}\cdot\frac{1}{9^5} + \&c.\right\} = 1{\cdot}609437892\ldots$$

$$\text{whence } \log_e 2 + \log_e 5 = \log_e 10 = 2{\cdot}302585066\ldots.$$

and hence $M = \dfrac{1}{\log_e 10} = \dfrac{1}{2{\cdot}302585066} = {\cdot}43429448\ldots$ the modulus.

16. PROP. *The difference between the logarithms of any two successive numbers diminishes as the numbers increase.*

Let u and $u + 1$ be any two successive numbers, m and $m + d$ their logarithms respectively.

Then $\log_{10}(u + 1) = m + d$, and $\log_{10} u = m$,

$$\text{and } (m + d) - m = \log_{10}(u + 1) - \log_{10} u$$

$$= \log_{10}\left\{\frac{u + 1}{u}\right\}$$

$$\therefore\ d = \log_{10}\left\{1 + \frac{1}{u}\right\}$$

Hence as u increases, $1 + \dfrac{1}{u}$ diminishes,

and $\therefore$ also $\log_{10}\left\{1 + \dfrac{1}{u}\right\}$ diminishes: consequently d also diminishes.

17. PROP. *To find the mantissa of the logarithm of a number consisting of six digits, having given the mantissa of the logarithm of the number composed of the first five digits, and the mantissa the logarithm of the next greater number.*[1]

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[1] Ex. 1. Given $\log_{10}43712 = 4{\cdot}6406007$, and $\log_{10}43713 = 4{\cdot}6406106$; to find $\log_{10}43712{\cdot}8$.

Generally $m = m_1 + (m_2 - m_1)\dfrac{a}{10}$.

Here $a = 8$, $m_2 = {\cdot}6406106$, $m_1 = {\cdot}6406007$; $\therefore\ m_2 - m_1 = {\cdot}0000099$;

and $(m_2 - m_1)\dfrac{a}{10} = {\cdot}0000099 \times \dfrac{8}{10} = {\cdot}0000079$, the increment for 8;

$\therefore\ m = m_1 + (m_2 - m_1)\dfrac{a}{10} = {\cdot}6406007 + {\cdot}0000079 = {\cdot}6406086$;

and since the number of integral places is 5, the characteristic is 4, and consequently $\log_{10} 43712{\cdot}8 = 4{\cdot}6406086$ the logarithm required.

Ex. 2. Conversely. Given $4{\cdot}6406007 = \log_{10}43712$, and $4{\cdot}6406106 = \log_{10}43713$, to find the number corresponding to the logarithm $4{\cdot}6406086$.

Generally $\dfrac{m - m_1}{m_2 - m_1} = \dfrac{a}{10}$.

$$\begin{array}{lrlcrl} \text{Here} & m = & {\cdot}6406086 & \qquad & m_2 = & {\cdot}6406106 \\ & m_1 = & {\cdot}6406007 & & m_1 = & {\cdot}6406007 \\ \hline & m - m_1 = & {\cdot}0000079 & & m_2 - m_1 = & {\cdot}0000099 \end{array}$$

$\therefore\ \dfrac{m - m_1}{m_2 - m_1} = \dfrac{{\cdot}0000079}{{\cdot}0000099} = \dfrac{8}{10} = {\cdot}8$ nearly, the sixth digit.

Hence $43712 + {\cdot}8 = 43712{\cdot}8$ is the number required.

Let m_1, m_2 be the mantissæ of u, $u+1$ respectively, and m the mantissa of $u+\frac{a}{10}$, a being the sixth digit.

Now since u, $u+1$ and $u+\frac{a}{10}$, are supposed to have the same number of integral places, the characteristics of these logarithms will be the same. Let this be c.

Then $c+m_1=\log_{10}u$, $c+m_2=\log_{10}(u+1)$,

$$\text{and } c+m=\log_{10}\left\{u+\frac{a}{10}\right\}.$$

$$\therefore\ m-m_1=\log_{10}\left\{u+\frac{a}{10}\right\}-\log_{10}u=\log_{10}\left\{1-\frac{a}{10u}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\log_e\left\{1+\frac{a}{10u}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\left\{\frac{a}{10u}-\frac{1}{2}\left(\frac{a}{10u}\right)^2+\text{\&c.}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\frac{a}{10u}.\ \text{nearly}\qquad(\text{I}).^1$$

$$\text{Again } m_2-m_1=\log_{10}(u+1)-\log_{10}u=\log_{10}\left\{1+\frac{1}{u}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\log_e\left\{1+\frac{1}{u}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\left\{\frac{1}{u}-\frac{1}{2u^2}+\frac{1}{3u^3}-\text{\&c.}\right\}$$

$$=\frac{1}{\log_e 10}\cdot\frac{1}{u}\ \text{nearly}\qquad(\text{II}).^1$$

$$\therefore\ \frac{m-m_1}{m_2-m_1}=\frac{a}{10},$$

---

[1] From the logarithm tables of numbers consisting of five digits, which are calculated to seven places of decimals, the logarithms of numbers consisting of six digits may be found correctly as far as the seventh place of decimals.

In the investigation, the second and all succeeding terms of the series for $m-m_1$ and m_2-m_1 have been omitted; and it must be shewn that these omissions do not affect the seventh place of decimals in the logarithms.

First to shew that the second term $\frac{1}{2\log_e 10}\cdot\left\{\frac{a}{10u}\right\}^2$ of the series for $m-m_1$ may be omitted.

$$\text{Since } \frac{1}{2\log_e 10}=\frac{1}{2\times 2{\cdot}302\ldots}=\frac{1}{4{\cdot}6\ldots}<\frac{1}{4{\cdot}5}\text{ or }\frac{2}{9},$$

and the least value of u is 10000, and the greatest value of $\frac{1}{u}$ is $\frac{1}{10000}$,

also a can never exceed 9.

$$\text{Therefore } \frac{1}{2\log_e 10}\cdot\left\{\frac{a}{10u}\right\}^2 \text{ is less than } \frac{2}{9}\times{\cdot}0000000081\text{ or }{\cdot}0000000018.$$

$$\text{And } \frac{1}{2\log_e 10}\cdot\frac{1}{u^2}\text{ is less than }\frac{2}{9}\times{\cdot}00000001\text{ or }{\cdot}00000000\dot{2}.$$

Whence, it appears that the second terms of the two series do not affect the seventh place of decimals of the logarithm, and therefore may be omitted, and *a fortiori* may the third and succeeding terms be omitted.

And $m = m_1 + (m_2 - m_1) . \frac{a}{10}$, of which $(m_2 - m_1) . \frac{a}{10}$ is the increment to be added to m_1; and it is called "the proportional part," for a sixth digit of the number, being a fourth proportional to 10, a and $m_2 - m_1$.

Cor. If the given number consisted of seven instead of six digits, a being the sixth and b the seventh digit:

then $m = m_1 + (m_2 - m_1) . \left\{ \frac{a}{10} + \frac{b}{10^2} \right\} = m_1 + (m_2 - m_1)\frac{a}{10} + \frac{1}{10}(m_2 - m_1)\frac{b}{10}$,

in which expression, $\frac{1}{10}(m_2 - m_1)\frac{b}{10}$ is the increment to be added for the seventh digit, which is one-tenth of what should be added, if b were the sixth digit.

18. Prop. *And conversely, to find the natural number corresponding to a given logarithm, having given the mantissa of the logarithm of the number composed of the first five digits, and the mantissa of the logarithm of the next greater number than that which is composed of its first five digits.*[1]

Employing the same notations, it has been proved that

$$\frac{m - m_1}{m_2 - m_1} = \frac{a}{10},$$

and since m, m_1, m_2, are given, it follows that $\frac{a}{10}$ is known from $\frac{m - m_1}{m_1 - m_2}$, and the sixth digit a is known, and $u + \frac{a}{10}$ is the number required.

To explain the method of constructing tables of proportional parts for a table of logarithms calculated for all numbers which do not exceed five digits; and their uses (1) in finding, by addition only, the logarithms of numbers consisting of one or two digits more than the numbers for which the tables are calculated; and (2) in finding, by subtraction only, the numbers corresponding to given logarithms not found exactly in the tables.[2]

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[1] See note on p. 13.

[2] Ex. 1. Find the logarithm of 43712·86.

$$\begin{array}{rl} \text{Here } \log_{10} 43712 = 4\cdot6406007 & \text{from the Tables} \\ 79 & = \text{prop. part for } 8 \\ 5,9 & = \quad ,, \quad ,, \quad 6 \\ \hline \therefore \log_{10} 43712\cdot86 = 4\cdot6406092 & \end{array}$$

Ex. 2. Conversely. Find the natural number corresponding to the logarithm 4·6406092.

$$\begin{array}{rl} \text{The given logarithm is } = 4\cdot6406092 & \\ \log_{10} 43712 = 4\cdot6406007 & \text{from the Tables} \\ \hline 85 & \\ 79 & = \text{prop. part for } 8 \\ \hline 60 & \\ 59 & \quad ,, \quad ,, \quad 6 \\ \hline 1 & \end{array}$$

Hence the number corresponding to the logarithm 4·6406092 is 43712·86.

If m_2 be the mantissa of the logarithm of a number of five digits next greater than, and m_1 of that next less than m the mantissa of a number whose sixth digit is a: it has been proved generally that the mantissa of the logarithm of the number of six digits is

$m = m_1 + (m_2 - m_1)\frac{a}{10}$, of which $(m_2 - m_1)\frac{a}{10}$ is the increment or the proportional part to be added to the mantissa of the logarithm of the next less number of five digits.

The method will be best shewn by an example.

To construct a table of proportional parts for the logarithms of the numbers between 43712 and 43713, consisting of six digits, having given $\log_{10} 43712 = 4{\cdot}6406007$ and $\log_{10} 43723 = 4{\cdot}6406106$.

Here $m_2 = {\cdot}6406106$, $m_1 = {\cdot}6406007$, $\therefore\ m_2 - m_1 = {\cdot}0000099$.

For a the sixth digit write 1, 2, 3, 4, 5, 6, 7, 8, 9, successively :—

a	$(m_2 - m_1)\frac{a}{10}$			
1	·00000099	=	·0000010	nearly
2	·00000198	=	·0000020	,,
3	·00000297	=	·0000030	,,
4	·00000396	=	·0000040	,,
5	·00000495	=	·0000050	,,
6	·00000594	=	·0000059	,,
7	·00000693	=	·0000069	,,
8	·00000792	=	·0000079	,,
9	·00000891	=	·0000089	,,

In the printed tables of logarithms in general use, the ciphers in the results above found are omitted, and only the significant figures of the difference of the logarithms and the proportional parts are registered on the right of each figure which may occupy the place of the sixth digit of the number. For the logarithms whose difference is 99, the table of proportional parts is given as here exhibited.

Since any digit in the seventh place of a number, is one-tenth of the same digit if it were in the sixth place, it follows that tables of proportional parts for digits in the seventh place will be one-tenth of what each would be in the sixth place; and consequently that separate tables for the proportioned parts for the seventh digits will be unnecessary.

D 99	
1	10
2	20
3	30
4	40
5	50
6	59
7	69
8	79
9	89

EXERCISES.

I.

1. What practical object had Napier in view when he made the discovery of logarithms? What improvement was suggested by Briggs?

2. Derive the word logarithm, and explain the terms of the equation $u = a^x$.

3. Explain and exemplify what is meant by a system of logarithms, and describe the two systems employed by mathematicians.

4. Find the numbers of which 2, 3, 4, 5, are the respective logarithms in that system whose base is 7.

5. Determine the logarithm of 32 to base 2, and to base 4; of 10000 to base 10; and of 20736 to base 12, and of 128 to base $2\sqrt{2}$.

6. What are the bases of the systems when the logarithm of 9 is equal to 2; the logarithm of 64 is equal to 12; the logarithm of 256 is equal to 8; and the logarithm of 2187 is equal to 7?

7. Explain the advantages of the base of the system of logarithms being the same number as the base of the denary system of notation.

8. What would be the difference between a system of logarithms computed to base 10, and another to base $\frac{1}{10}$?

II.

Having given $\log_{10}2$, $\log_{10}3$, $\log_{10}7$, $\log_{10}11$, $\log_{10}13$, $\log_{10}17$, $\log_{10}19$.

Calculate the logarithms of the following numbers:—$\log_{10}5$, $\log_{10}125$, $\log_{10}484$, $\log_{10}504$, $\log_{10}441$, $\log_{10}3240$, $\log_{10}30$, $\log_{10}1200$, $\log_{10}64000$, $\log_{10}14700$, $\log_{10}111475$, $\log_{10}{\cdot}03$, $\log_{10}{\cdot}0012$, $\log_{10}{\cdot}00064$.

III.

1. Find the values of $\log_{10}{\cdot}\dot{3}$, $\log_{10}{\cdot}2\dot{3}$, $\log_{10}{\cdot}41\dot{6}$, and $\log_{10}{\cdot}\dot{4}2857\dot{1}$.

2. Determine $\log_3 10$, $\log_8 10$, $\log_5{\cdot}01$, $\log_5 10$, and $\log_{12}1800$.

3. Given $\log_{10}16$, $\log_{10}7$, $\log_{10}27$, find $\log_{10}15$, $\log_{10}56$, $\log_{10}1764$.

4. Given $\log_{10}15 = 1{\cdot}176091$, $\log_{10}21 = 1{\cdot}322219$, and $\log_{10}35 = 1{\cdot}544068$; to find the logarithms of 3, 5, 7.

5. Given $\log_{10}2$, find the values of $\log_{10}({\cdot}0625)^{-1}$, $\log_{10}({\cdot}0125)^{-\frac{1}{3}}$.

6. Given $\log_{10}7 = {\cdot}8450980$, $\log_{10}9{\cdot}824394 = {\cdot}9923057$, find $({\cdot}00007)^{\frac{1}{540}}$.

7. Find $\log_{10}27$ and $\log_{10}4\frac{1}{20}$ in terms of $\log_{10}15$ and $\log_{10}16$; and conversely, given $\log_{10}27$, and $\log_{10}4\frac{1}{20}$, find $\log_{10}15$ and $\log_{10}16$.

8. Given $\log_{10}8 = {\cdot}90309$ and $\log_{10}9 = {\cdot}9542425$, find $\log_{10}15$ and $\log_{10}6$; and conversely, given $\log_{10}15$ and $\log_{10}6$, find $\log_{10}8$ and $\log_{10}9$.

IV.

Shew the truth of the following equivalents respectively :—

1. $\log_{10}(1+2+3)=\log_{10}1+\log_{10}2+\log_{10}3$.
2. $7.\log_{10}\frac{16}{15}+5.\log_{10}\frac{25}{24}+3.\log_{10}\frac{81}{80}=\log_{10}2$.
3. $7.\log_{10}\frac{11}{16}+6.\log_{10}\frac{8}{3}+5.\log_{10}\frac{2}{5}+\log_{10}\frac{32}{25}=\log_{10}3$.
4. $16.\log_{10}\frac{16}{15}+12.\log_{10}\frac{25}{24}+7.\log_{10}\frac{81}{80}=\log_{10}5$.

V.

Prove the truth of the following expressions and find the logarithms of each. $\text{Log}_{10}139=2{\cdot}1430148$.

1. $\log_{10}(\frac{2}{3})^{11}=11\{\log_{10}2-\log_{10}3\}$.
2. $\log_{10}(\frac{63}{20})^{\frac{1}{3}}=\frac{1}{3}\{2\log_{10}3+\log_{10}7-\log_{10}2-1\}$.
3. $\log_{10}(\frac{23}{417})^{\frac{3}{5}}=\frac{3}{5}\{\log_{10}23-\log_{10}3-\log_{10}139\}$.
4. $\log_{10}\left\{\dfrac{10.(\cdot045)^{\frac{1}{3}}}{(7{\cdot}20)^{\frac{1}{5}}}\right\}=\frac{1}{15}\{2+4\log_{10}3-14\log_{10}2\}$.
5. $\log_{10}\left\{\dfrac{(32)^{\frac{1}{4}}\times(48)^{\frac{1}{3}}}{2(27)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}=\frac{1}{24}\{19\log_{10}2-14\log_{10}3\}$.
6. $\log_{10}\left\{\dfrac{\{3(60^2-15^2)60\}^{\frac{1}{3}}}{\{(60+15)\times(16\times9)^{\frac{1}{2}}\}^{\frac{1}{2}}}\right\}=\frac{1}{3}\{1+2(\log_{10}3-\log_{10}2)\}$.
7. $\log_{10}\left\{\dfrac{\{5.2^{\frac{1}{2}}\times2.5^{\frac{1}{3}}\}^{\frac{1}{4}}}{10^{\frac{1}{2}}\times2^{\frac{1}{3}}}\right\}=\frac{1}{24}\{4+7\log_{10}2\}$.

VI.

Find the numerical values of x in each of the following equations, having given $\log_{10}2$, $\log_{10}3$, and $\log_{10}17$:—

$5^x=20:\ 100^x=2:\ 8^x=100:\ 25^x=800:\ 8^{7x+1}.2^{6x+3}=5^{4x+3}.20^{3x+2}:$
$3^x=15:\ 12^x=180:\ 2^{3x}.3^{2x}=10:\ 15^{2x}=24^{x+1}:\ 10^x=102:$
$2^{x+5}+2.2^{x+4}=2^{12}:\ 8^x.125^{2-x}=2^{4x+3}.5^x:\ 5^{1-4x}=2^{4x+1}.$

VII.

Solve the following equations :—

$ac^x-bd^x=0:\ a^{mx}=b-a^{mx-1}:\ a^{mx}.b^{nx}=c:\ m^{nx+c}-m^{nx-c}=n:$
$a^x.c^{nx}=b^{mx-n}:\ a^{3x+2}.b^{5x+3}=c^{x-1}.d^{4x-5}:2^{\frac{1}{2}}+1-(2^{\frac{1}{x}}-1)^{-1}=0:$
$(a+b)^2(a^4-2a^2b^2+b^4)^{x-1}=(a-b)^{2x}:\ a^{x^2+1}=b^{x^2-1}:$
$\left\{\dfrac{a}{c}\right\}^{x-2m}=\left\{\dfrac{c}{a}\right\}^{3x}:\ \dfrac{a^{mx}}{b^{nx}}=(c^x+dc^{x+n}).$

VIII.

Find the values of x and y in each of the following sets of equations :—

$a^xb^y=p,\ c^xd^y=q:\qquad a^{mx}.b^{ny}=r,\ c^{px}.d^{qy}=s:\qquad a^{x+y}=b,\ b^x=a^y:$
$x^{x+y}=y^n,\ y^{x+y}=x:\ x^{x+y}=y^{4n},\ y^{x+y}=x^n:\ a^{2x}-b^{2y}=m^2,\ a^x-b^y=n:$
$(ax)^{\log a}=(cy)^{\log c},\quad c^{\log x}-a^{\log y}:\ x^y=y^x,\ x^m=y^n.$

IX.

Determine the values of x, y, z, in the following equations:—

1. $\alpha^x . \beta^y . \gamma^z = a, \quad \alpha^y . \beta^z . \gamma^x = b, \quad \alpha^z . \beta^x . \gamma^y = c.$

2. $a^x . b^{y+z} = c, \quad a_1^{\,y} . b_1^{\,x+z} = c_1, \quad a_2^{\,z} . b_2^{\,x+y} = c_2.$

3. $y^{\log z} . z^{\log y} = a^2, \quad z^{\log x} . x^{\log z} = b^2, \quad x^{\log y} . y^{\log x} = c^2.$

X.

Find the values of the unknown quantity in each of the quadratic equations:—

$a^{2x} + ba^x + c = 0 : \; a^x + a^{-x} = c : \; a^x + a^{2x} = a^{3x} : \; a^x - 8a^{-x} = 2 :$

$\dfrac{\log a}{a^x} + \dfrac{a^x}{\log a} = 2 : \; a^{x - \frac{r}{x}} - b^{x-p} . c^{qx} = 0 : \; \left\{ \dfrac{a^m}{x} \right\}^{\log_e x} = x^{\log_e b} :$

$4^{2x} - 8.4^x + 7 = 0 : \; 3^{2x} + 3^x = 4785156 : \; 4^{x+2} + 16^x = 5120 :$

$10^x - 60.10^{-x} - 4 = 0 : \; 5^{2x} = 100.5^{x-3} + 5^{x-1} : \; 27(3^{x+1} - 3^{2x+3}) = 2 :$

$3^{2x} - 2.3^{x+1} = 567 : \; 2^{2x} - 100.2^{x+1} = 14336 : \; 3^{x+4} - 3^{2x+6} = 2.$

$\log_e \left\{ \dfrac{\sqrt{a^2+x^2}+a}{\sqrt{a^2+x^2}-a} \right\} = b : \log_{10} \left\{ \dfrac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right\} = 2 : \; \dfrac{1+\sqrt{1-e^{2x}}}{1-\sqrt{1-e^{2x}}} = (ce^x)^2.$

XI.

1. Prove that x^x is greater than e, and $x^{\frac{1}{x}}$ less than e, when x is any positive integer greater than 2.

2. If x and y be positive, prove that x^y is less than y^x when x is greater than y, and y greater than e: but x^y is greater than y^x when x is greater than y, and y less than e.

3. Solve the equation $x^x = 100$ by approximation to three places of decimals.

4. Find x and y in whole numbers from the equation $x^y = y^x$.

5. Solve the equations $x^y = 500$ and $y^x = 300$, by approximation.

6. Find the values of x in each of the four following equations:—

$a^{e^x} = m : \; c^{a^x} = n : \; a^{x^e} = p : \; a^{(\log_e x)^n} = m^{\log_e m}.$

7. Express the equation $m(\log_e a + x) = n \log_e x + 1$ in an exponental form.

8. Find x and y in the three following sets of equations:—

$2^{x^y} = 512, \; 2^{y^x} = 256 : \; (2^{2^x})^{2y} = 16, \; \{3^{3^{\frac{1}{x}}}\}^{-\frac{1}{3y}} = 3 : \; (4^x)^y = 16, \; 3^x . 9^{\frac{y}{2}} = 27.$

9. Shew whether $\log_5\{\log_{10} 25\}$ is equivalent to $\log_{10}\{\log_5 25\}$.

10. Find x^x from $x^{3x} - 21x^{2x} + 147x^x - 316 = 0$.

XII.

1. If $y = a^x$, and x, a and $\log_e y$ be in harmonical progression, find the values of x and y.

2. Given $\log_8 3 = m$, and $\log_{25} 24 = n$; find $\log_{10} 45$.

3. Find $\log_e x$ from the equation $\dfrac{\log_e x + a}{\log_{10} x} - \dfrac{\log_{10} x - b}{\log_e x} = c$.

4. In the equation $\dfrac{\log_e x - 1}{\log_{10} x} + \dfrac{2\log_{10} x + 3}{4\log_e x} = 1$, $\log_{10} x = \dfrac{4(\log_e 10 - 1) + 1}{(2\log_e 10 - 1)^2 + 1}$.

5. The common logarithm of one number is greater by unity than the logarithm of another, and the logarithm of their difference is one-half of the logarithm of their sum. Find the numbers.

6. The area of a right angled triangle whose hypothenuse is x^{3x} and sides x^{2x} and x^x, is $\frac{1}{2}\sqrt{2+\sqrt{5}}$.

XIII.

1. Define what is meant by the *characteristic* and *mantissa* of a logarithm. Shew that the characteristics of the logarithms of numbers to base 10 may always be determined by inspection, and that the mantissa depends on the sequence of figures only, and not on the position of the decimal point.

2. Explain why the mantissa of a logarithm is always in calculation considered positive, while the characteristic may be positive or negative.

3. Given $\log_{10}81{\cdot}123 = {\cdot}9096256$, write the logarithms of 81123, ·81123, ·0081123, 8112300, and 8112·3 respectively.

4. Find the characteristics of the logarithms of 300, ·003, and 3·2 to base 5; of 81 to base 3; of 81 to base $\sqrt{3}$; of 25 to base 100; of ·003 to base 8.

5. Having given $\log_{10}2 = {\cdot}30103$, $\log_{10}3 = {\cdot}47712$, $\log_{10}7 = {\cdot}84509$; find the characteristics of $\log_{12}1200$, $\log_{10}7^{-1}$; $\log_4\sqrt{300}$; $\log_{\sqrt{3}}49$: $\log_{1000}\left\{\frac{4}{343}\right\}^{\frac{1}{5}}$.

6. Find the number of digits in 2^{63}, 125^{100}, 1000^{1000}, e^{1000} respectively, having given $\log_{10}2 = {\cdot}3010300$ and $\log_{10}e = {\cdot}4342956$.

7. If $5^{-100000}$ be expressed as a decimal, how many ciphers will there be before the first significant digit of the decimal?

8. Find the integral values between which x must lie, in order that the integral part of $(1{\cdot}08)^x$ may contain 4 digits.

9. How many times must the number 2 be multiplied into itself, so that the product may just exceed 1,000,000,000?

10. Given p the characteristic of $\log_a N$, what is the characteristic of $\log_x N$, when $x = a^q$?

11. If the characteristics of the logarithms of any numbers be known, find the limits between which the characteristic of the logarithm of their product will lie; (1) if all the numbers be greater than unity, and (2) if a certain number of them be less than unity.

XIV.

1. What is meant by the *modulus* of any two systems of logarithms? Find a general expression for the modulus which connects any two systems of logarithms, and apply it to find the moduli which connect the systems whose bases are 9 and 10; 10 and 20; and 5 and 7 respectively.

2. Prove that the ratio $\frac{\log_e x}{\log_a x}$ is a constant quantity, and incom-

mensurable; and shew how its value may be determined to any degree of accuracy when $a=10$.

3. Assuming $\log_y x = \log_y z \,.\, \log_z x$,
thence shew that $\log_a b \,.\, \log_b a = 1$, and $\log_b a \,.\, \log_c b \,.\, \log_a c = 1$.

4. Shew that the exponental expressions a^x, x^y, y^x may be put under the forms $e^{x\log_c a}$, $e^{y\log_e x}$, $e^{x\log_e y}$ respectively.

5. If $e^{\log_a x} = b \,.\, e^{\log_c x}$, find an expression for b.

6. Prove the truth of the three following expressions:—

$$n^{\log_a m} = m^{\log_a n} : \ \frac{\log_b n}{\log_a n} = \frac{\log_x a}{\log_x b} : \ \frac{\log_a x}{\log_a y} = \frac{\log_b x}{\log_b y}.$$

7. Verify the following equivalents:

$$(ab)^{\log a + \log b} = a^{\log a} .\, b^{\log b} .\, a^{2\log b}.$$

$$\text{and } (ab)^{\log a - \log b} = (ab^{-1})^{\log a + \log b} = a^{\log a} .\, b^{-\log b}.$$

8. If $y^q = z^p$, prove that $p \log_y a = q \log_z a$.

9. Shew that $\log_c\{p^{\log_a n}\} = \log_a n \,.\, \log_a p \,.\, \log_c b \,.\, \log_b a$.

XV.

1. Prove that $\log_e a$ lies between $n(1 - a^{-\frac{1}{n}})$ and $n(a^{\frac{1}{n}} - 1)$, when n is assumed of such a value that $a^{\frac{1}{n}}$ is very nearly equal to unity.

2. Prove $\dfrac{1}{e} = \dfrac{1}{1.3} + \dfrac{1}{1.2.3.5} + \dfrac{1}{1.2.3.4.5.7} + \ldots\ldots = \cdot 36788 \ldots\ldots$

3. Shew that the value of $(1 + \frac{1}{x})^x = 2.7182 \ldots$ when x is indefinitely increased.

4. Shew that $\log_e(1 + x)$ is less than x for all positive values of x, whether greater or less than unity.

5. If x be very small, shew that $e^{e^x} = e(1 + x)$ nearly.

6. Shew that $\log_e 101 - \log_e 99 = \frac{1}{50}$ very nearly.

7. Prove that $\log_e x = n(x^{\frac{1}{n}} - 1)$ nearly, when n is very great; and thence shew that $\log_e x = (x - 1) .\, \dfrac{2}{x^{\frac{1}{2}} + 1} .\, \dfrac{2}{x^{\frac{1}{4}} + 1} .\, \dfrac{2}{x^{\frac{1}{8}} + 1} \ldots$

8. Shew that $\left(1 + \dfrac{x}{n}\right)^n = e^x\left(1 - \dfrac{x^2}{2n}\right)$ nearly when n is large, and find the next term of the series of which the function on the second side is the commencement.

9. Prove that $\log_{10}\left\{1 + \dfrac{1}{10^{n+1}}\right\}$ is less than $\dfrac{1}{9 \,.\, 10^n}$ and exemplify its truth when n is 3 and 4.

XVI.

1. If $e^x = y + \sqrt{1 + y^2}$, then $y = \frac{1}{2}(e^x - e^{-x})$ and $\sqrt{1 + y^2} = \frac{1}{2}(e^x + e^{-x})$; and shew for what value of y, e^x is less than 2.

2. If $\dfrac{(a^2 - b^2)^x}{a + b} = 1$, shew that $\dfrac{x}{1 - x} = \dfrac{\log_e(a + b)}{\log_e(a - b)}$.

3. If c, s be the sums to infinity and to n terms of a decreasing geometrical progression whose first term is a;
shew that $n \log\left\{1-\frac{a}{c}\right\} = \log\left\{1-\frac{s}{c}\right\}$.

4. If a, b, c be the p^{th}, q^{th}, and r^{th} terms respectively, both of an arithmetical, and of a geometrical series, shew that

$$(1).\ (b-c)\log a + (c-a)\log b + (a-b)\log c = 0.$$

$$(2).\ \frac{q-r}{b\log c - c\log b} = \frac{r-p}{c\log a - a\log c} = \frac{p-q}{a\log b - b\log a}.$$

5. If $(\log_e y)^x = x$:

shew that $\dfrac{y - y(\log_e y)^x . \log_e(\log_e y)}{x(\log_e y)^{x-1}} = e^{e^{\frac{\log x}{x}}} . e^{\frac{\log x}{x}} . \dfrac{1-\log x}{x^2}$.

6. If $\log_a x$, $\log_b y$, $\log_c z$ be in arithmetical progression having unity for the common difference, shew that

$$ac = b^{\frac{\log_e x}{\log_e y - \log_e b} + \frac{\log_e z}{\log_e y + \log_e b}}.$$

7. If $\dfrac{x(y+z-x)}{\log x} = \dfrac{y(z+x-y)}{\log y} = \dfrac{z(x+y-z)}{\log z}$;

prove that $y^z . z^y = z^x . x^z = x^y y^x$.

8. Prove $\dfrac{\log_a(\log_a N)}{\sqrt{\log_a b}} - \dfrac{\log_b(\log_b N)}{\sqrt{\log_b a}} = \dfrac{\log_a(\log_a b)}{\sqrt{\log_a b}} = -\dfrac{\log_b(\log_b a)}{\sqrt{\log_b a}}$.

XVII.

Shew the truth of each of the following series for $\log_e x$:—

1. $(x-x^{-1}) - \frac{1}{2}(x^2-x^{-2}) + \frac{1}{3}(x^3-x^{-3}) - \&c.$
2. $\frac{1}{m}\{(x^m-1) - \frac{1}{2}(x^m-1)^2 + \frac{1}{3}(x^m-1)^3 - \&c.\}$
3. $-m\{(1-x^{\frac{1}{m}}) + \frac{1}{2}(1-x^{\frac{1}{m}})^2 + \frac{1}{3}(1-x^{\frac{1}{m}})^3 + \&c.\}$
4. $2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \&c.\right\}.$
5. $\frac{1}{2}\log_e(x+1) + \frac{1}{2}\log_e(x-1) - \left\{\frac{1}{2x^2-1} + \frac{1}{3(2x^2-1)^3} + \&c.\right\}.$

XVIII.

1. Shew that the base of Napier's system of logarithms differs from the sum of the first $n+1$ terms of the series

$$1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1.2.3} + \&c.,$$

by a quantity less than $\dfrac{1}{1.2.3.4\ldots(n-1).n^2}$.

2. Having given $\log_e 2 = {\cdot}6931472$; show that $\log_e 3 = 1{\cdot}0986123$.

3. Given $\log_e 4 = 1{\cdot}38629$; find $\log_e 5$ to five places of decimals.

4. Having given $\log_e 2 = 0{\cdot}6931$, $\log_e 10 = 2{\cdot}3026$; find $\log_e 5$, and $\log_e 11$.

5. Given $\log_e 32 = 3{\cdot}4657360$ and $\log_e 5 = 1{\cdot}6094379$; find $\log_{10} 2$.

6. Calculate $\log_{10} 99$ to four places of decimals, having given $\log_e 10 = 2{\cdot}3026$ nearly.

7. Given $\log_{10}12 = 1{\cdot}0791812$, $\log_{10}1{\cdot}121563 = {\cdot}0498256$, $\log_{10}1{\cdot}257915 = {\cdot}0996512$; find the value of $\{(1{\cdot}44)^{-6} - (1{\cdot}44)^{-12}\} \div {\cdot}12$.

8. Solve the equation $10^x = 101$ having given $\log_e 10 = 2{\cdot}30258$.

XIX.

1. If a, b, c be three consecutive numbers, prove that

$$\log_e b = \tfrac{1}{2}\log_e c + \tfrac{1}{2}\log_e a - \left\{\frac{1}{2ac+1} + \frac{1}{3(2ac+1)^3} + \frac{1}{5(2ac+1)^5} + \&c.\right\}$$

and explain the advantage of the use of this formula in calculating the logarithms of prime numbers. Determine how many terms of the series must be taken that the logarithms may be correct for seven places of decimals, when the numbers consist of 2, 3, 4 or 7 digits.

2. Find the coefficient of x^n in the expansions of $(1+x)^n.e^x$, and of $(1-x)^n.\,e^{-x}$.

3. If A_{2n-1}, A_{2n} be the coefficients of x^{2n-1}, x^{2n}, in the expansion of $\log_e\{(1+x)^{\frac{1}{1-x}}\}$, according to the ascending powers of x;

shew that $2n\,(A_{2n-1} - A_{2n}) = 1$.

4. $$\log_e(1+x)^{\frac{x+1}{2}} + \log_e(1-x)^{\frac{x-1}{2}} = \frac{x^2}{1\,.\,2} + \frac{x^4}{3\,.\,4} + \frac{x^6}{5\,.\,6} + \&c$$

5. Shew that $x = e^{\frac{1}{1-\log z}}$ arises from the elimination of y between the equations $y = e^{\frac{1}{1-\log x}}$ and $z = e^{\frac{1}{1-\log y}}$:

and if $(\log x)^r + (\log y)^r + (\log z)^r = S_r$,

prove that $3 + S_2 + S_3 + S_4 + \ldots\ldots = 0$.

6. $$\log_a(n^{\log_a n}) = \frac{\frac{1}{2}(n-1)^2 - \frac{1}{3}(1+\frac{1}{2})(n-1)^3 + \frac{1}{4}(1+\frac{1}{2}+\frac{1}{3})(n-1)^4 - \&c.}{\frac{1}{2}(a-1)^2 - \frac{1}{3}(1+\frac{1}{2})(a-1)^3 + \frac{1}{4}(1+\frac{1}{2}+\frac{1}{3})(a-1)^4 - \&c.}$$

XX.

1. Point out any advantages derived from the use of e as the base for logarithms in analytical calculations, and of 10 as the base in arithmetical calculations.

2. Explain the terms, finite and infinite, definite and indefinite, and shew that though each of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \&c.$, and $1 + \frac{1}{3} + \frac{1}{5} + \&c.$ continued indefinitely, be infinite, their difference is finite, and equal to $\log_e 2$.

3. Prove that $$\left(1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\right)^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \cdots$$

4. $\text{Log}_e\left(\dfrac{1}{1-x}\right) = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots$ ad infin.: hence calculate $\log_{10}4$ to 5 places of decimals, having given $\log_{10}e = {\cdot}43429$.

5. Expand $\log_e(x^2 + 5x + 6)$ in a series of descending powers of x.

6. $$\text{Log}\,(x+n) = \log n + \log\left(1 + \frac{1}{x}\right) + \log\left(1 + \frac{1}{1+x}\right) + \;\ldots\; + \;\ldots\; \log\left(1 + \frac{1}{n-1+x}\right).$$

7. If S_2, S_3, S_4, ... S_n be the sums to 2, 3, 4 .... n terms of any series $A + B + C + D + \ldots$; shew that

$$\log S_n = \log S_2 + \log\left(1 + \frac{C}{S_2}\right) + \log\left(1 + \frac{D}{S_3}\right) + \log\left(1 + \frac{E}{S_4}\right) + \text{\&c.}$$

XXI.

1. What multiple of x must z be, that a^z may be equal to e^x?

2. If a, b, c be in geometrical progression, shew that $\log_c a$, $\log_b c$, and $\log_a b$ are in arithmetical progression and their common difference is $\frac{3}{2}$.

3. If $\log_a b_1 = \log_b b_2 = \log_c b_3 =$ &c., where a, b, c, &c., are in geometrical progression, show that b_1, b_2, b_3, &c., will also be in geometrical progression.

4. If $a^x = b^y = c^z =$ &c., and x, y, z, &c., be in harmonical progression, then shall a, b, c, &c., be in geometrical progression: and conversely, if a, b, c, &c., be in geometrical progression, then shall x, y, z, &c., be in harmonical progression.

5. If $\log_x a$, $\log_x b$, $\log_x c$ .... be in arithmetical progression, shew that $\log_a y$, $\log_b y$, $\log_c y$ .... are in harmonical progression.

6. Shew that if the bases of different systems of logarithms increase in geometrical progression, their moduli will decrease in harmonical progression.

XXII.

1. By means of the expansion of e^x, show that

$$e^{x\sqrt{-1}} + e^{-x\sqrt{-1}} = 2\left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} - \text{\&c.}\right\}$$

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = 2\sqrt{-1}\left\{x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \text{\&c.}\right\}$$

2. Shew that whatever be the value of x, the series

$$(x - x^{-1}) - \tfrac{1}{3}(x^3 - x^{-3}) + \tfrac{1}{5}(x^5 - x^{-5}) + \text{\&c.} = \log_e \sqrt{-1}.$$

3. Express $\log_e (x + y\sqrt{-1})$ in the form of $a + b\sqrt{-1}$.

4. The imaginary part of $\log_e(a + b\sqrt{-1})^{\sqrt{-1}}$ is $\sqrt{-1}\log_e\sqrt{a^2 + b^2}$.

5. Find the value of $\log_e\{(1 + x)(1 + \alpha x)(1 + \beta x)\}$ in a series where α, β are the two impossible cube roots of unity.

6. Shew that $\log_e(1 + x + x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} -$ &c.

7. Expand $\log_e\left(\frac{1 + x + x^2}{1 - x + x^2}\right)$ in a series of ascending powers of x.

XXIII.

1. Shew how far the error committed in obtaining the logarithm of a number of six digits from the logarithms of numbers of five digits by means of proportional parts, can affect the first seven decimal places of the required logarithm.

2. Explain the method of constructing the tables of proportional parts for a table of logarithms. Given $\log_{10} 22713 = 4{\cdot}3562745$,

$\log_{10}22714 = 4\cdot3562936$; construct such a table for calculating the logarithms of numbers between 227130 and 227140.

3. If $\log_{10}67833 = 4\cdot8314410$, and $\log_{10}67832 = 4\cdot8314346$; find $\log_{10}67832800$.

4. Given $\log_{10}71820 = 4\cdot8562454$, and $\log_{10}71821 = 4\cdot8562514$; find $\log_{10}\cdot07182035$.

5. Having given $\log_{10}33819 = 4\cdot5291608$, and
$\log_{10}33818 = 4\cdot5291479$; find the logarithm of $338\cdot185$.

6. Having given $\log_{10}71968 = 4\cdot8571394$; diff. for $1 = 60$: find the value of $\log_{10}(\cdot0719686)^{\frac{1}{3}}$ to seven places of decimals.

7. Find the numerical value of $\log_{10}(\cdot242447)^{14}$, having given $\log_{10}24244 = 4\cdot3846043$; diff. for $1 = 179$.

XXIV.

1. The difference of successive integral numbers being invariable, shew that as those numbers increase, the difference of their logarithms diminishes: also that the difference of the logarithms of two consecutive numbers n and $n+1$ is less than $\frac{1}{n}$ in the Napierian system, and less than $\frac{1}{2n}$ in the common system.

2. Shew how to find the number corresponding to a given logarithm when it is not found exactly in the tables.

Given $\log_{10}37649 = 4\cdot5757534$, and $\log_{10}37650 = 4\cdot5757650$; find the number corresponding to the logarithm $1\cdot575638$.

3. Given $\log_{10}1\cdot0686 = \cdot02881517$, and $\log_{10}1\cdot0687 = \cdot02885581$; find the number of which the logarithm is $\cdot02883549$.

4. If $\log_{10}4\cdot3125 = \cdot6347291$, and $\log_{10}4\cdot3126 = \cdot6347392$; determine the number whose logarithm is $\bar{3}\cdot6347362$.

5. Given that e is $2\cdot7182818$, and that $\log_{10}2\cdot7182 = \cdot43428147$ and $\log_{10}2\cdot7183 = \cdot4342974$; find $\log_{10}999995$ correctly to twelve places.

6. Approximate to the value of $5^{\sqrt{5}}$, having given $\log_{10}2 = \cdot30103$, $\log_{10}349485 = 5.543428$, $\log_{10}1\cdot562944 = \cdot193943$, $\log 3\cdot655 = \cdot562887$, $\log_{10}3\cdot656 = \cdot563006$.

7. Find the value of $10^{10^{\frac{1}{3}}}$, having given $\log_{10}21544 = 4\cdot3333268$,
$\log_{10}21545 = 4\cdot3333465$, $\log_{10}14270 = 4\cdot1544240$, $\log_{10}14271 = 4\cdot154$.

8. Given $9^{3^x} = 1000$, $\log_{10}2 = \cdot30103$, $\log_{10}300 = 2\cdot4771213$, $\log_{10}\cdot47712 = \bar{1}\cdot6786276$, $\log\cdot47713 = \bar{1}\cdot6786367$; shew that the value of x lies between $1\frac{1}{23}$ and $1\frac{1}{24}$.

9. Could the results of calculation be depended on, which are made by a system of logarithms where the value of the base cannot be exactly determined?

10. Was Napier led to the discovery of logarithms by geometrical or numerical considerations? Explain his method, and compare it with the method by which Newton was led to the discovery of fluxions.

RESULTS, HINTS, ETC., FOR THE EXERCISES.

I.

1. See Section V., pp. 4, 5. 2. Section V., p. 6, and Section XII., Art. 1. 3. Art. 1, note. 4. Since $7^2=49$, $7^3=343$, $7^4=2401$, $7^5=16807$, 2, 3, 4, 5 are the respective logarithms of 49, 343, 2401, 16807, to base 7. 5. Since $2^5=32$, $4^x=2^5$, $10^4=10000$, $12^4=20736$, $2^{\frac{3x}{2}}=2^7$; 5 is the logarithm of 32 to base 2, $2\frac{1}{2}$ of 32 to base 4, 4 of 10000 to base 10, 4 of 20736 to base 12, and $4\frac{2}{3}$ of 128 to base $2\sqrt{2}$.

6. Let x denote the base in each instance, then $x^2=3^2$, $x^{12}=2^6$, $x^8=2^8$, $x^7=3^7$, and the respective bases are 3, $2^{\frac{1}{2}}$, 2, 3.

7. See Art. 6, 7. 8. $\frac{1}{10}$ being the same as 10^{-1}, suggests the answer.

II.

Express the given numbers in their prime factors, or in quotients which involve 10 or powers of 10; and employ the logarithms of the prime numbers between 1 and 100 in the note p. 4, and the law of the characteristics under Art. 7, exemplified in the note. The results are—

$\log 5=\cdot6989700$: $\log 125=2\cdot0969100$: $\log 484=2\cdot6848454$: $\log 504=2\cdot7024305$:
$\log 441=2\cdot6444386$: $\log 3240=3\cdot5105450$: $\log 30=1\cdot4771213$:
$\log 1200=3\cdot0791812$: $\log 64000=4\cdot8061800$: $\log 14700=4\cdot1673173$:
$\log 111475=5\cdot0471774$: $\log \cdot03=\bar{2}\cdot4771213$: $\log \cdot0012=\bar{4}\cdot0791812$:
$\log \cdot00064=\bar{5}\cdot8061800$.

III.

1. $\text{Log}\ \cdot\dot{3}=\log(\frac{1}{3})=\log 1-\log 3=-\log 3$: $\log \cdot2\dot{3}=\log(\frac{7}{30})=\log 7-\log 3-1$: $\log\cdot41\dot{6}=\log(\frac{5}{12})=\log 5-2\log 2-\log 3$: $\log\cdot\dot{4}2857\dot{1}=\log(\frac{3}{7})=\log 3-\log 7$.

2. Let $\log_3 10=x$, then $3^x=10$, and $x\log 3=1$, whence x is known. The other four examples may be done in the same way.

3. $\text{Log}\ 16=\log 2^4=4\log 2$, $\therefore \log 2=\frac{1}{4}\log 16$, $\log 27=\log 3^3=3\log 3$:

$$\therefore \log 3=\frac{1}{8}\log 27.$$

$$\text{Then } \log 15=\log 3+\log 5=\log 3+1-\log 2=\frac{1}{3}\log 27-\frac{1}{4}\log 16+1.$$

$$\log 56=3\log 2+\log 7=\frac{3}{4}\log 16+\log 7 :$$

$$\log 1764=\log(2^2.3^2.7^2)=2\log 2+2\log 3+2\log 7=\frac{1}{2}\log 16+\frac{2}{3}\log 27+2\log 7.$$

4. Since $\log 15=\log 3+\log 5$, $\log 21=\log 3+\log 7$, $\log 35=\log 5+\log 7$, from these three equations $\log 3$, $\log 5$, $\log 7$ can be readily found.

5. $\text{Log}\,(\cdot0625)^{-1}=\log\left(\frac{1}{\cdot0625}\right)=\log\left(\frac{10^4}{5^4}\right)=\log 2^4=4\log 2.$

$$\log(\cdot0125)^{-\frac{1}{3}}=\log\left(\frac{10^4}{5^3}\right)^{\frac{1}{3}}=\log(10.2^3)^{\frac{1}{3}}=\frac{1}{3}+\log 2.$$

6. $\log(\cdot00007)^{\frac{1}{540}}=\frac{1}{540}\log\cdot00007=\frac{\bar{5}\cdot8450980}{540}=\bar{1}\cdot9923057=\log\cdot09824394.$

$$\text{Hence since } \log(\cdot00007)^{\frac{1}{540}}=\log(\cdot09824394)$$

$$\text{therefore } (\cdot00007)^{\frac{1}{540}}=\cdot09824394.$$

7. First $\log 16=4\log 2$, and $\log 2=\frac{1}{4}\log 16$.

$$\text{Next } \log 15=\log\tfrac{30}{2}=1+\log 3-\log 2, \text{ and } \log 3=\log 15-1-\tfrac{1}{4}\log 16.$$

$$\text{Then } \log 27=3\log 3=3\,(\log 15-1-\tfrac{1}{4}\log 16)$$

$$\text{and } \log 4\tfrac{1}{20}=\log(\tfrac{81}{20})=4\log 3-1-\log 2=4\log 15+\tfrac{3}{4}\log 16-5.$$

Conversely,

$$\log 27 = 3 \log 3, \text{ and } \log 3 = \tfrac{1}{3} \log 27$$
$$\log 4\tfrac{1}{20} = 4 \log 3 - 1 - \log 2, \text{ and } \log 2 = \tfrac{4}{3} \log 27 - 1 - \log 4\tfrac{1}{20}.$$
$$\text{Then } \log 16 = 4 \log 2 = 4\left\{\tfrac{4}{3} \log 27 - 1 - \log 4\tfrac{1}{20}\right\}$$
$$\text{and } \log 15 = 1 + \log 3 - \log 2 = 2 + \log 4\tfrac{1}{20} - \log 27.$$

8. As the preceding.

IV.

These equivalents are shown to be true by Arts. 2, 3, 4.

V.

1. $\bar{2}\cdot 0629957$. 2. $\cdot 1661035$. 3. $\bar{1}\cdot 2449388$. 4. $\bar{1}\cdot 9796043$. 5. $\bar{1}\cdot 9599946$. 6. $\cdot 4507275$. 7. $\cdot 2544670$.

VI.

If the logarithms of the two members of each equation be taken, the values of x may be found by help of the logarithms of the prime numbers given in the note to Art. 5, p. 4.

VII.

By taking the logarithms of the members of these equations the values of the unknown quantities may be found in terms of the quantities supposed to be of known values.

$\log a + x \log c = \log b + x \log d$: $mx \log (1 + a^{-1}) = \log b$: $mx \log a + nx \log b = \log c$: $mx \log (m_c - m^{-c}) = \log n$: $x \log a + nx \log c = (mx - n) \log b$:
$(3x+2) \log a + (5x+3) \log b = (x-1) \log c + (4x-5) \log d$: $x = 2$:
$x \log (a+b) = \log (a-b)$: $x^2(\log b - \log a) = \log a + \log b$: $x = \frac{m}{2}$.
$mx \log a = nx \log b + x \log c + \log (1 + dc^n)$.

VIII.

Here $x \log a + y \log b = \log p$, $x \log c + y \log d = \log q$, are two equations of the first degree, with two unknown quantities. The same remark applies to the next two equations: Here $a^{x+y} = a^x a^y = a^x b^x = b$. By equating values of x, $(x+y)^2 = n$, and $x = y = 1$. By equating values of x or y, $(x+y)^2 = 4n^2$ and $x = y^2$, from which x and y may be found. Divide the first by the second equation. Taking the logarithms of each equation, $(\log a)^2 + \log a . \log x = (\log c)^2 + \log c . \log y$ and $\log x \log c = \log a \log y$, by substitution $\log x$ and $\log y$ may be determined.

$$x = \left(\frac{m}{n}\right)^{\frac{n}{m-n}}, \text{ and } y = \left(\frac{m}{n}\right)^{\frac{m}{m-n}}.$$

IX.

1. Take the logarithms of each equation, and there results three equations with three unknown quantities.

2. The same remark applies to the sets 2 and 3.

X.

From the first equation a^x may be found as from an ordinary quadratic. The next six equations when reduced are $a^{2x} - ca^x = -1$, $a^{2x} - a^x = 1$, $a^{2x} - 2a^x = 8$, $a^{2x} - 2a^x \log a = (\log a)^2$, $x^2 (\log a - \log b - 9 \log c) + px \log b = r \log a$,

$$\log_e x(m \log_e a - \log_e x - \log_e b) = 0.$$

$4^x = 7$ and 1. $3^x = 2187$ and -2188. $4^x = 64$ and -80. $10^x = 10$ and -6. The reduced equation is $5^{2x} - 5^x = 0$, $3^3 . 3^x = 2$ and 1. $3^x - 27$ and -21. $2^x = 256$ and -56.

The equation $\log_e \left\{\dfrac{\sqrt{a^2 + x^2} + a}{\sqrt{a^2 + x^2} - a}\right\} = b$, which expressed in an exponential form is $\frac{\sqrt{x^2+x^2}+a}{\sqrt{x^2+x^2}-a}=e^b$; from which $x=o$ and $\frac{2ae^{\bar{2}}}{e^b-1}$. $x=o$ and $\frac{20}{101}$. The equation reduced is $c^2e^{2x}-2c+1=0$.

XI.

1. The base e of Napier's logarithms is greater than 2, but less than 3. Let $x=2$, thus $x^x=2^2=4$, which is greater than e; and so a fortiori is x^x greater than e for any value of x greater than 2. In like manner $x^{\frac{1}{x}}$ is shown to be less than e.

2. First, if x be greater than y, and y greater than e; let $x=4$, $y=3$, then $x^y=4^3=64$, and $y^x=3^x=81$, and x^y is less than y^x, in this particular case.

Next if x be greater than y and y be less than e; let $y=2$, $x=3$, then $x^y=3^2=9$ and $y^x=2^3=8$, and x^y is greater than y^x, in this case.

The results will be the same for greater numbers, subject to the given conditions.

3. There is no direct method known for the solution of exponential equations of this form. Different methods of trial have been attempted, of which the following has the preference.

Taking the logarithms of both members of the equation $x^x=100$, $x\log x=2$. The question is thus reduced to finding a number such that the product of the number required and its logarithm shall be equal to 2. On trial, it will be found that x is greater than 3 but less than 4.

If $x=3$, $3\log 3=1{\cdot}4313639$; and if $x=4$, $4\log 3=2{\cdot}4082400$. And the arithmetic mean between these two results being very nearly equal to 2, it follows that the true value of x is nearer to 4 than to 3.

If $x=3{\cdot}5$, $3{\cdot}5\log 3{\cdot}5=1{\cdot}9042386$; and if $x=3{\cdot}6$, $3{\cdot}6\log 3{\cdot}6=2{\cdot}0026890$. Hence the true value of x lies between 3·5 and 3·6.

Since the difference is small, if the increments of the logarithms may be taken proportional to the increments of the numbers (as in the proportional parts for tables of logarithms), a correction may be found for the error in the assumed value of x.

$$\begin{array}{l} 2{\cdot}0026090 \\ \underline{1{\cdot}9042386} \\ {\cdot}0984504 \end{array}$$
·0984504 increment of logarithm for ·1, difference of 3·6 and 3.5.

$$\begin{array}{l} 2{\cdot}0026090 \\ \underline{2.\qquad\qquad} \\ {\cdot}0026090 \end{array}$$
·0026090 increment of logarithm for y, difference of 3·6 and x the required value.

$$\text{Thus } \frac{y}{{\cdot}1}=\frac{{\cdot}0026890}{{\cdot}0984504}={\cdot}027303, \text{ and } y={\cdot}0027303.$$

Hence $x=3{\cdot}6-{\cdot}0027303=3{\cdot}5972697$.

On trial this value of x will be found too small.

A nearer approximation may be found by a similar process, taking 3·59727 and 3·59728 instead of 3·5 and 3·6, and a second correction may be found, which will give a result correct to a greater number of decimal places.

4. The equation is indeterminate, and admits of an indefinite number of solutions. Let $y=nx$, then $x^{nx}=(nx)^x$, and $x^{n-1}=n$. If n be successively taken equal to 2, 3, 4, &c., a series of the value of x will be found, and from $y=nx$, a corresponding series of the values of y will be known. The only solution in whole numbers is that in which $x=2$, $y=4$.

5. See the solution of question 3 above.

6. Let $y=e^x$, in the equation $a^{e^x}=m$, then $a^y=m$; and taking the logarithms $\log_e y=x$, $y\log_e a=\log_e m$, and $\log_e y=-\log_e(\log_e a)+\log_e(\log_e m)$. Hence $x=\log_e y=\log_e(\log_e m)-\log_e(\log_e a)$.

By a similar process the other three equations can be solved.

7. $x^n = a^m . e^{mx+1}$.

8. The *first two* equations are reducible to $x^y = 9$ and $y^x = 8$, which give integral values of x and y.

By eliminating y between *the second two* equations, $x^2 \log 2 - x \log 6 + \log 3 = 0$.

From the *third set* of two equations $xy = 2$ and $x + y = 3$.

9. Converting the two expressions into their equivalent exponential forms. Thus, if $\log_5(\log_{10} 25) = x$ and $\log_{10} 25 = z$ then $\log_5 z = x$: and $10^z = 25$, also $5^x = z$, whence $10^{5x} = 25$. Also if $y = \log_{10}(\log_5 25)$, then $5^{10^y} = 25$.

If $10^5 = 5^{10}$, the expressions are equivalent.

10. The equation may be put into the form $(x^x)^3 - 21(x^x)^2 + 147(x^x) - 316 = 0$. Let $x^x = z$, then the equation becomes $z^3 - 21 z^2 + 147z - 316 = 0$, an ordinary equation of the third degree, which has one integral root equal to 4. The equation may be reduced to a quadratic and the other two roots can be found. The values of x^x are 4 and $\frac{1}{2}\{17 \pm \sqrt{210}\}$.

XII.

1. The condition leads to the equation $2 \log_e a = a (\log_e a + 1)$.

2. $\text{Log}_{10} 45 = \dfrac{3(m+1) + 12mn}{3(m+1) + 2n}$.

3. $\text{Log}_e x = -\log_e 10 \{a \log_e 10 + b\} \div \{(\log_e 10)^2 - c \log_e 10 - 1\}$.

5. The question gives the equations $\dfrac{x}{y} = 10$ and $(x-y)^2 = (x+y)$.

6. By Euc. I. 47, $x^{4x} + x^{2x} = x^{6x}$, whence can be found the two sides containing the right angle.

XIII.

For questions 1, 2, 3, see Arts. 6, 7, 8 and the notes.

4. To find the characteristic of the logarithm of 300 to base 5. Let $\log_5 300 = x$, then $5^x = 300$ and $x \log 5 = \log 300$, from which $x = 3{\cdot}5\ldots.$ The characteristic is 3, which also appears from the fact that 300 in the denary scale is equal to 2220 in the quinary.

5. These may be found as in the preceding. Let $\log_{12} 12000 = x$, then $12^x = 12000$, and $x \log 12 = \log 12000$, whence $x = 4{\cdot}2\ldots$: the characteristic is 4, and so for the rest.

6. As the characteristic of the logarithm of a number is always less by unity than the number of integral digits of which that number partly or wholly consists; this property can be employed to determine the number of digits in the high powers of numbers.

Since $\log_{10} 2^{63} = 63 \log_{10} 2 = 18{\cdot}96489$, the characteristic being 18, the number of digits in 2^{63} is 19.

Similarly the number of digits in 125^{100} is 210; in 1001^{1000} is 3001; and in e^{1000} is 435.

7. In this example the characteristic will be negative. See Art. 7.

8. Since 1000 is the least and 9999 the greatest number of 4 digits, the values of x determined from the equations $(1{\cdot}08)^x = 1000$ and $(1{\cdot}01)^x = 9999$, will suggest the limits required, $x = 89$ and 119.

9. If x denote the number, $2^x = 10^9$, and $x \log 2 = 9$ whence $x = 29{\cdot}897\ldots$ which indicates that 2 must be multiplied 30 times, and $2^{30} = 1,065,741,824$.

10. Here $\log_a N = p + m$, m being the mantissa, $\log_x N = \dfrac{\log_e N}{\log_e x}$, and here $x = a^q$, $\log_e x = q \log_e a$, it follows that $\log_x N = \dfrac{\log_e N}{q \log_e a} = \dfrac{\log_a N}{q} = \dfrac{p}{q} + \dfrac{m}{q}$.. The characteristic is $\dfrac{p}{q}$.

11. See Arts. 7, 8.

XIV.

1. See Arts. 9, 10, 15. 2. See Arts. 9, 12, 15. See Art. 9.

4. If $(x \log_e a)$ be substituted for x in the series for e^x, the series becomes identical to the series for a^x.

5. See Art. 9.

6. Let $\log_a m = x$, $\log_a x = y$; then $a^x = m$ and $a^y = n \because (a^y)^x = (a^x)^y$. The other two equations may be deduced from $\log_a u . \log_c a = \log_c u$. Art. 9. Or thus

$$\log_a m . \log_e n = \log_a n . \log_e m, \quad \frac{\log_a m}{\log_e m} = \frac{\log_a n}{\log_e n} = \frac{1}{\log_e a}.$$

7. In the verification, the first of the three equations in question 6 is required. In these verifications, it must be remembered that $a^{\log b} = b^{\log a}$.

8. Since $y^q = z^p$, then $q \log_a y = p \log_a z$: but $\log_a y . \log_y a = 1$ and $\log_a z . \log_z a = 1$.

$\therefore \log_a y . \log_y a = \log_a z . \log_z a$. Whence $\dfrac{q}{\log_y a} = \dfrac{p}{\log_z a}$, and $q \log_z a = p \log_y a$.

9. $\text{Log}_c \{ p^{\log_a n} \} = \log_a n . \log_c p$, $\log_c p = \dfrac{\log_a p}{\log_a c}$, $\log_a c . \log_b a . \log_c b = 1$, whence $\log_c \{ p^{\log_a n} \} = \log_a n . \log_a p . \log_a b . \log_b a$.

XV.

1. Generally $\log_e u = (u-1) - \dfrac{(u-1)^2}{2} + \dfrac{(u-1)^3}{3} - \&c.$

$$\text{Let } u = a^{\frac{1}{n}}, \text{ then } \log_e a = n \left\{ (a^{\frac{1}{n}} - 1) - \frac{(a^{\frac{1}{n}} - 1)^2}{2} + \frac{(a^{\frac{1}{n}} - 1)^3}{3} - \&c. \right\} \quad (1)$$

$$\text{and let } u = a^{-\frac{1}{n}}, \text{ then } \log_e a = n \left\{ (1 - a^{-\frac{1}{n}}) + \frac{(1 - a^{-\frac{1}{n}})^2}{2} + \frac{(1 - x^{-\frac{1}{n}})^3}{3} + \&c. \right\} \quad (2)$$

in both which series n is any number whatever, and a is a positive integer.

In series (1), $a^{\frac{1}{n}}$ can always be found, so that $a^{\frac{1}{n}} - 1$ shall be a very small fraction: and since the aggregates of the second and third terms, the fourth and fifth, &c., are always negative, it follows that $\log_e a$ is less than $n(a^{\frac{1}{n}} - 1)$.

In series (2), $1 - a^{-\frac{1}{n}}$ is a still smaller fraction, but as all the terms of this series are positive, it follows that $\log_e a$ is greater than $n(1 - a^{-\frac{1}{n}})$.

Hence $\log_e a$ lies between $n(a^{\frac{1}{n}} - 1)$ and $n(1 - a^{-\frac{1}{n}})$.

2. This is deduced from the expansion of e^x, by taking $x = -1$.

3. $$(1 + \tfrac{1}{x})^x = 1 + \frac{x}{1} + \frac{x(x-1)}{1.2} . \frac{1}{x^2} + \frac{x(x-1)(x-2)}{1.2.3} . \frac{1}{x^3} + \&c.$$
$$= 1 + 1 + \frac{1.(1 - \frac{1}{x})}{1.2} + \frac{1.(1 - \frac{1}{x})(1 - \frac{2}{x})}{1.2.3} . + \&c.$$
$$= 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c., \text{ when } x \text{ is indefinitely increased.}$$

4. That is, the Napierian logarithm of any number increased by unity is always less than the number assumed.

5. Since x is very small $e^x = 1 + x + \dfrac{x^2}{1.2} + \&c. = 1 + x$ nearly.

$$\text{and } e^{e^x} = e^{1+x} = e . e^x = e \left\{ 1 + x + \frac{x^2}{1.2} + \&c. \right\} = e(1+x) \text{ nearly.}$$

6. $$\log_e \left(\frac{101}{99} \right) = \log_e \left\{ 1 + \frac{2}{99} \right\} = \frac{2}{99} - \frac{1}{2} \left(\frac{2}{99} \right)^2 + \&c. = \frac{2}{99} \text{ nearly} = \frac{1}{50} \text{ nearly.}$$

7. Since x is very great, as x increases, $x^{\frac{1}{x}}$ decreases; and when x becomes indefinitely great, x may be considered equivalent to an indefinitely great power of 2.

Also $x - 1 = (x^{\frac{1}{2}} - 1)(x^{\frac{1}{2}} + 1)$, $x^{\frac{1}{2}} - 1 = (x^{\frac{1}{4}} - 1)(x^{\frac{1}{4}} + 1)$, &c., whence the expression for $\log_e x$ may be deduced.

8. When $\left(1+\frac{x}{n}\right)^n$ is expanded according to the form in Art. 11,

$$\left(1+\frac{x}{n}\right)^n=\left\{1+x+\frac{x^2}{1.2}+\&c.\right\}-\frac{x^2}{2n}\left\{1+x+\frac{x^2}{1.2}+\&c.\right\}+\frac{1}{n^2}\left\{\frac{x^3}{3}+\&c.\right\}$$
$$=e^x-\frac{x^2e^x}{2n}+\frac{1}{n^2}\left\{\frac{x^3}{3}+\&c.\right\}$$
$$=e^x\left\{1-\frac{x^2}{2n}\right\}+\frac{1}{n^2}\left\{\frac{x^3}{3}+\&c.\right\}$$

XVI.

1. Transpose y to the left side of the equation and square each side. If $e^x=2$, then $y=\frac{3}{4}$; therefore y must be less than $\frac{3}{4}$ in order that e^x may be less than 2.

2. First find the value of x, next the value of $1-x$.

3. Let x be the common ratio; then $c=\frac{a}{1-x}$, and $\frac{c}{a}=\frac{1}{1-x}$, also $\frac{a}{c}=1-x$, $1-\frac{a}{c}=x$

$$\text{and } s=\frac{a(1-x^n)}{1-x},\ \frac{s}{a}=\frac{1-x^n}{1-x}\ \therefore\ 1-x^n=\frac{s}{a}\,.\,(1-x)=\frac{s}{a}\times\frac{a}{c}=\frac{s}{c},\ 1-\frac{s}{c}=x^n.$$

$$\text{Hence } 1-\frac{s}{c}=\left(1-\frac{a}{c}\right)^n,\ \text{and } \log\left(1-\frac{s}{c}\right)=n\log\left(1-\frac{a}{c}\right).$$

4. (1) Let x, y be respectively the first terms and common difference of the arithmetic series, and the first term and common ratio of the geometric series.

$$\text{Then } x+(p-1)\,.\,y=a,\ \ x+(q-1)y=b,\ \ x+(r-1)y=c:$$
$$\text{also } xy^{p-1}=a,\ \ xy^{q-1}=b,\ xy^{r-1}=c.$$

From these equations respectively, both expressions may be found.

5. First $(\log_e y)^x=x$, then $\log_e y=x^{\frac{1}{x}}$, and $\log_e(\log_e y)=\frac{\log_e x}{x}$, also $y=e^{x^{\frac{1}{x}}}$.

$$\text{next } (\log_e y)^x=x,\ \log_e y=x^{\frac{1}{x}},\ (\log_e y)^{x-1}=x^{\frac{x-1}{x}}.$$

6. See Art. 9, and eliminate $\log_a x$, $\log_b y$, and $\log_c z$.

7. First, taking the first and second expressions for the first equation. Clearing from fractions, it becomes $x\log y(y+z-x)=y\log x(z+x-y)$,

$$\text{whence may be found } \frac{\log y^{xz}}{\log x^{yz}}=\frac{\log(x^yy^x)^x}{\log(x^yy^x)^y}. \quad (1)$$

Next, taking the second and third expressions for the second equation, and reducing as before, there results

$$\frac{\log z^{xy}}{\log y^{xz}}=\frac{\log(z^yy^z)^y}{\log(z^yy^z)^z}. \quad (2)$$

Thirdly, taking the first and third expressions for the third equation, and from it results

$$\frac{\log x^{zy}}{\log z^{xy}}=\frac{\log(z^xx^z)^z}{\log(z^xx^z)^x}. \quad (3)$$

Multiplying these three resulting equations

$$\frac{\log y^{xz}}{\log x^{yz}}\cdot\frac{\log z^{xy}}{\log y^{xz}}\cdot\frac{\log x^{zy}}{\log z^{xy}}=\frac{\log(x^yy^x)^x}{\log(x^yy^x)^y}\cdot\frac{\log(z^yy^z)^y}{\log(z^yy^z)^z}\cdot\frac{\log(z^xx^z)^z}{\log(z^xx^z)^x}.$$

$$\text{or } \frac{\log(y^xx^y)^x}{\log(z^xx^z)^x}\cdot\frac{\log(z^yy^z)^y}{\log(y^xx^y)^y}\cdot\frac{\log(x^zz^x)^z}{\log(z^yy^z)^z}=1.$$

$$\text{Hence } \frac{\log(y^xx^y)^x}{\log(z^xx^z)^x}=1,\ \ \frac{\log(z^yy^z)^y}{\log(y^xx^y)^y}=1,\ \text{and } \frac{\log(x^zz^x)^z}{\log(z^yy^z)^z}=1.$$

$$\text{Therefore } \log(y^xx^y)^x=\log(z^xx^z)^x,\ \ (y^xx^y)^x=(z^xx^z)^x,\ \text{and } y^xx^y=z^xx^z;$$
$$\text{Similarly } \log(x^yy^z)^y=\log(y^xx^y)^y,\ \ (x^yy^z)^y=(y^xx^y)^y,\ \text{and } x^yy^z=y^xx^y;$$
$$\text{and } \log(x^zz^x)^z=\log(z^yy^z)^z,\ \ (x^zz^x)^z=(z^yy^z)^z,\ \text{and } x^zz^x=z^yy^z.$$
$$\text{Wherefore } x^yy^x=y^zz^x=x^zz^x.$$

8. See Art. 9, and assume $\log_a N=x$, and $\log_b N=y$.